VORTEX COLLAPSE FOR THE L^2-CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. The focusing cubic nonlinear Schrödinger equation in two dimensions admits vortex solitons, standing wave solutions with spatial structure, $Q^{(m)}(r,\theta) = e^{im\theta}R^{(m)}(r)$. In the case of spin $m = 1$, we prove there exists a class of data that collapse with the vortex soliton profile at the log-log rate. This extends the work of Merle and Raphaël, (the case $m = 0$,) and suggests that the L^2 mass that may be concentrated at a point during generic collapse may be unbounded. Difficulties with $m \geq 2$ or when breaking the spin symmetry are discussed.

Contents

1. Introduction

We consider the L^2-critical nonlinear Schrödinger equation in two dimensions,

$$\begin{cases} iu_t + \Delta u + u|u|^2 = 0 \\ u(0,x) = u_0 \in H^1(\mathbb{R}^2). \end{cases} \tag{1.1}$$

Equation (1.1) is locally wellposed for data $u_0 \in H^1$, [7, 11]. That is, there exists a solution $u \in C\left([0,T_{\max}), H^1\right)$ and some fixed negative power so that $T_{\max} \geq T_{lwp} = \|u_0\|_{H^1}^{-C}$. Therefore, we have the classic blowup alternative,

$$T_{\max} = +\infty \quad \text{or,} \quad \lim_{t \to T_{\max}} \|u(t)\|_{H^1} = +\infty.$$

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Evolution of u_0 by equation (1.1) preserves the following quantities.

$$M[u_0] = M[u(t)] = \int_{\mathbb{R}^2} |u(t,x)|^2 \, dx, \qquad \text{(mass)} \tag{1.2}$$

$$E[u_0] = E[u(t)] = \int |\nabla_x u(t,x)|^2 \, dx - \frac{1}{2} \int |u(t,x)|^4 \, dx, \qquad \text{(energy)} \tag{1.3}$$

$$P[u_0] = P[u(t)] = \mathrm{Im} \left(\int \overline{u}(t,x) \nabla u(t,x) \, dx \right). \qquad \text{(momentum)} \tag{1.4}$$

The associated symmetries of the equation are phase, time translation, and spatial translation. There is a Galilean symmetry,

$$u_{\beta_0}(t,x) = u(t, x - \beta_0 t) e^{i \frac{\beta_0}{2} \cdot (x - \frac{\beta_0}{2} t)}, \quad \text{for any fixed } \beta_0 \in \mathbb{R}^2,$$

and a scaling symmetry,

$$u_{\lambda_0}(t,x) = \lambda_0 u(\lambda_0^2 t, \lambda_0 x) \quad \text{for any fixed } \lambda_0 > 0.$$

The effect of scaling on Sobolev norms is, $\|u_{\lambda_0}\|_{\dot{H}^s} = \lambda_0^{-s} \|u\|_{\dot{H}^s}$, for any reasonable s. Note that only the critical norm is left invariant. By choosing $\lambda_0 = \|u(t)\|_{H^1}$ at a fixed time, and using the minimum local wellposedness time for unit data in H^1, we have the scaling lower bound for the blowup speed,

$$u(t) \in C\left([0, T_{\max}), H^1\right), \text{ with } T_{\max} \text{ maximal, then } \quad \|u(t)\|_{H^1} \gtrsim \frac{1}{\sqrt{T_{\max} - t}}.$$

Alternatively, the scaling lower bound can be established through energy conservation, [2].

Peculiar to the L^2-critical case, there is also the pseudo-conformal (or lens) symmetry,

$$v(t,x) = \frac{1}{T-t} u \left(\frac{1}{(T-t)^2}, \frac{x}{T-t} \right) e^{-i \frac{|x|^2}{4(T-t)}}, \tag{1.5}$$

which acts on the virial space, $\left\{ f \in H^1 \right\} \cap \left\{ |x|^2 f \in L^2 \right\}$. In particular, the pseudo-conformal symmetry transforms standing wave solutions into blowup solutions with H^1 norm growth $\frac{1}{T-t}$.

1.1. **Blowup with Soliton Profile.** To find standing wave solutions of equation (1.1), introduce the usual ansatz, $u(t,x) = e^{it} Q(x)$, to derive the profile equation,

$$\Delta Q - Q + Q|Q|^2 = 0. \tag{1.6}$$

There is a unique real-valued positive radial solution Q to equation (1.6), as proved by McLeod and Serrin [16][1]. This solution we call the soliton, or the ground-state since $E(Q) = 0$. In this paper we will focus on other solutions of equation (1.6), as we discuss in the next section. Weinstein [31] identified the soliton as the unique minimizer of $J[f] = \frac{|\nabla f|_{L^2}^2 |f|_{L^2}^2}{|f|_{L^4}^4}$ among H^1 functions, thereby showing the optimal constant of the Gagliardo-Nirenberg inequality,

$$\|v\|_{L^4}^4 \leq \frac{2}{\|Q\|_{L^2}^2} \|v\|_{\dot{H}^1}^2 \|v\|_{L^2}^2.$$

Note that if $M[u_0] < M[Q]$, the Gagliardo-Nirenberg inequality gives apriori control of the H^1 norm from the conservation of energy. That is, there is global wellposedness for data with $M[u_0] < M[Q]$.

The pseudo-conformal transformation (1.5) applied to the standing wave solution $e^{it}Q(x)$ gives an explicit blowup solution with $M[u_0] = M[Q]$. We denote this explicit solution $S(t)$; Merle [17] showed that, up to symmetries, it is the only blowup solution with the mass of Q. Bourgain and Wang [1] proved that $S(t)$ is stable with respect to perturbations that are exceptionally flat near the central profile.

More generally, negative energy data in the virial space leads to blowup, as shown by Glassey's virial identity [8],

$$\frac{d^2}{dt^2} \int |x|^2 |u(t)|^2 = 4 \frac{d}{dt} Im \int x \cdot \nabla u \overline{u} = 16 E[u_0]$$

Ogawa and Tsutsumi [25] later extended the argument to negative energy radial data.

[1]Following earlier work by Coffman [3] in 3D. Kwong [12] extended the result to all H^1-subcritical nonlinearities.

Let us consider $\mathcal{B}_\alpha = \{u_0 \in H^1 : M[Q] < M[u_0] < M[Q] + \alpha\}$, where $\alpha > 0$ is some small constant. Merle and Raphaël [19] proved that there is no solution in $\mathcal{B}_\alpha$ that blows up as predicted by Glassey's virial identity[2]. They also showed [18, 22] that there is an open subset $\mathcal{O} \subset \mathcal{B}_\alpha$, including all the negative energy data, that lead to blowup in finite time with the log-log rate,

$$\|u(t)\|_{H^1} \approx \sqrt{\frac{\log|\log(T-t)|}{T-t}}.$$

Raphaël [27] proved that all solutions in $\mathcal{B}_\alpha$ that lead to blowup either belong to $\mathcal{O}$, or blowup with at least the H^1 growth rate of $S(t)$. Finally, Merle and Raphaël [20] showed that all solutions in $\mathcal{B}_\alpha$ that blowup concentrate exactly the profile Q at a point, in the sense that there are parameters $\lambda(t) > 0$, $\gamma(t) \in \mathbb{R}$ and $\overline{x}(t) \in \mathbb{R}^2$ such that,

$$u(t,x) - \frac{1}{\lambda(t)} Q\left(\frac{x - \overline{x}(t)}{\lambda(t)}\right) e^{-i\gamma(t)} \longrightarrow u^*(x),$$

where the convergence is in L^2 as $t \to T_{\max}$. Moreover, the residual profile u^* identifies the blowup regime, with $u^* \notin H^1$ if and only if the solution belonged to $\mathcal{O}$ and followed the log-log rate.

1.2. **Vortex Solitons.** Vortex solitons are solutions to equation (1.6) of the form $Q^{(m)}(r,\theta) = e^{im\theta} R^{(m)}(r)$, where $R^{(m)}$ is real-valued and positive. That is, we seek a function $R^{(m)}$ that satisfies,

$$\begin{cases} \Delta R^{(m)} - \left(1 + \dfrac{m^2}{r^2}\right) R^{(m)} + \left(R^{(m)}\right)^3 = 0, \\ \partial_r R^{(m)}|_{r=0} = 0, \quad R^{(m)}(|x|) > 0, \quad R^{(m)} \in H^1(\mathbb{R}^2) \cap \left\{|x|^{-1} f(x) \in L^2\right\} \end{cases} \tag{1.7}$$

For all $m \in \mathbb{Z}$, Iaia and Warchall [9] showed there exists a solution to (1.7) and, analogous to the result of Kwong [12] in the case $m = 0$, Mizumachi [23] has shown it is unique. Fibich and Gavish [4, Lemma 12] have remarked that the resulting profile Q_m is the unique minimizer of $J[f] = \frac{|\nabla f|_{L^2}^2 |f|_{L^2}^2}{|f|_{L^4}^4}$ among H^1 functions with spin m. We denote this space by $H^1_{(m)}$. Some vortex solutions are pictured in Figure 1, and their radial profiles appear in Figure 2.

This variational characterization gives an optimal Gagliardo-Nirenberg inequality for functions in $H^1_{(m)}$. As a consequence, for data $u_0 \in H^1_{(m)}$ and L^2 norm less than $\|Q^{(m)}\|_{L^2}$ there is global wellposedness. As a second consequence, Fibich and Gavish [4, Corollary 16] remark that $\|Q^{(m)}\|_{L^2}^2$ is a strictly increasing sequence in m. Indeed, Pego and Warchall [26] showed the asymptotic form,

$$R^{(m)}(r) \approx \left(1 + \frac{m^2}{r_{\max}^2}\right)^{\frac{1}{2}} \sqrt{2}\operatorname{sech}\left(\left(1 + \frac{m^2}{r_{\max}^2}\right)^{\frac{1}{2}} (r - r_{\max})\right),$$

where $r_{\max} \approx \sqrt{2}m$ for $m \gg 0$. Therefore, $\|Q^{(m)}\|_{L^2}^2 \approx 4\sqrt{3}m$ for large m, which Fibich and Gavish found to be a good approximation[3] even for small m.

The linearization of equation (1.1) near $Q^{(m)}$ is,

$$\partial_t v = -iL^{(m)}[v], \quad \text{where,} \quad L^{(m)}[v] \equiv \left(-\Delta + 1 - |Q^{(m)}|^2\right) v - 2\left(Q^{(m)}\right)^2 \overline{v}. \tag{1.8}$$

Written as a harmonic series, $v = \sum_{j\in\mathbb{Z}} e^{i(m+j)\theta} f_j(r)$,

$$L^{(m)}[v] = \sum_j \left(-\Delta + 1 - |Q^{(m)}|^2\right) e^{i(m+j)\theta} f_j - 2|Q^{(m)}|^2 e^{i(m-j)\theta} \overline{f_j}, \tag{1.9}$$

so that it is clear the linear system excites harmonics in pairs. In the case involving only $j = 0$, that is, $v = e^{im\theta}(v_1 + iv_2)$, we may write $-iL^{(m)}[v]$ in matrix form as,

$$\begin{bmatrix} 0 & L_-^{(m)} \\ -L_+^{(m)} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{where,} \quad \begin{aligned} L_+^{(m)} &= -\Delta + 1 - 3|Q^{(m)}|^2, \\ L_-^{(m)} &= -\Delta + 1 - |Q^{(m)}|^2. \end{aligned} \tag{1.10}$$

[2]There is no solution in $\mathcal{B}_\alpha$ for which $\lim_{t\to T_{\max}} \int |x|^2 |u(t)|^2 = 0$, in constrast to the explicit solution $S(t)$.

[3]Error less than 3% for $m = 2$, less than 0.4% for $m \geq 5$.

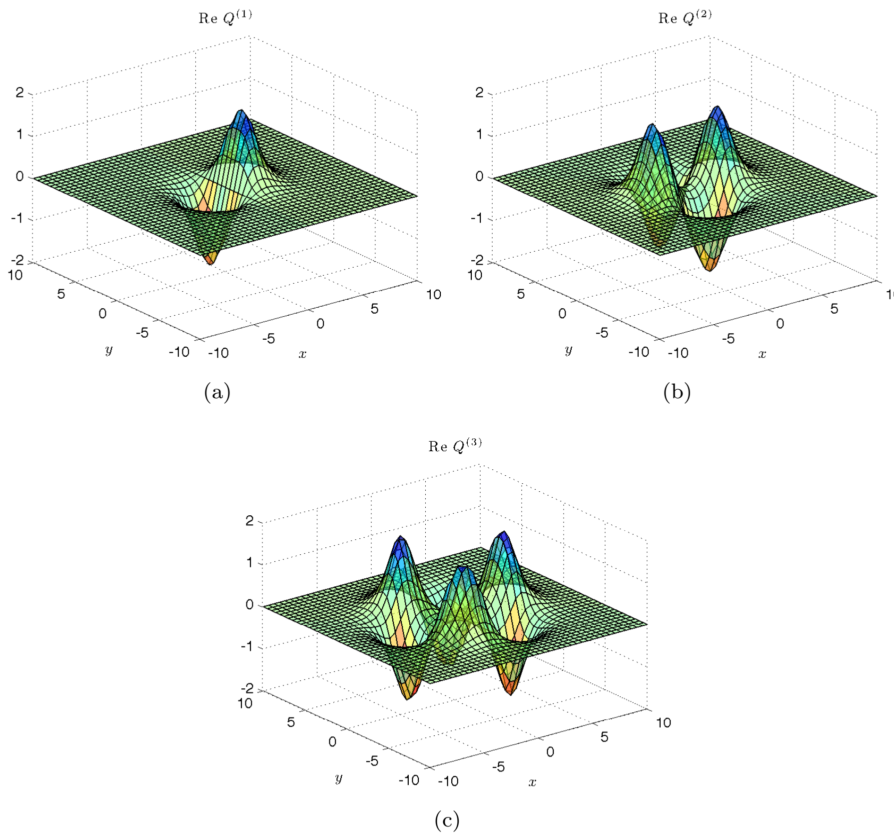

FIGURE 1. The real component of some vortex solutions.

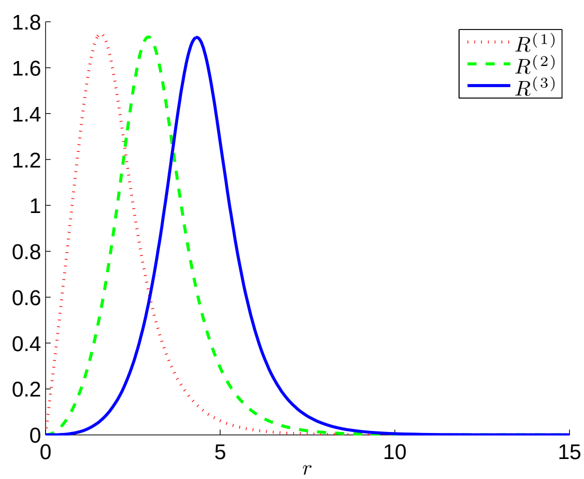

FIGURE 2. The radial component of some vortex solutions.

Comparing equations (1.9) and (1.10) we see that $L^{(m)}$ takes on the form of (1.10) on all of H^1 in the case of spin $m = 0$. In this important case, Weinstein [32] showed that the generalized nullspace of L has dimension 8 and is generated by the symmetries.

In the cases $m = 1$ and $m = 2$, the generalized nullspace of $L^{(m)}$ is generated in the same way. However, in these cases Pego and Warchall [26] found unstable eigenvalues and additional eigenvalues in the spectral

gap (all for modes with $|j| \neq 0$). That is, there exists a function ρ with spin $m = 1$ such that,

$$L_+^{(m)}(\rho) = -|y|^2 Q_m, \qquad L_-^{(m)}\left(|y|^2 Q_m\right) = -4\Lambda Q_m,$$
$$L_+^{(m)}(\Lambda Q_m) = -2Q_m, \quad L_-^{(m)}(Q_m) = 0,$$

where $\Lambda = 1 + y \cdot \nabla$ denotes the scaling operator. The remaining Jordan chains, generated by $\nabla Q^{(m)}$, consist of functions with $|j| = 1$. Instability of vortex profiles is not restricted to the cubic nonlinearity. Mizumachi [24] has shown that there are unstable vortex profiles for any power-type nonlinearity strictly stronger than linear.

1.3. **Blowup with Vortex Profiles.** Any vortex soliton becomes a blowup solution through the pseudo-conformal transformation. Study of the asymptotic profile during vortex blowup was initiated by Fibich & Gavish [4], including the variational structure referenced above. Their work includes numerical simulations where they found data with mass slightly larger than $Q^{(m)}$ that blowup at exactly the scaling lower bound and with profiles different from the vortex soliton. [4]

Our main result is that there is a class of solutions with spin $m = 1$ that blowup with exactly the vortex soliton profile and log-log behaviour similar to that established in the case $m = 0$.

Theorem 1.1 (Log-log Blowup with Vortex Profile)**.** *Assume the Spectral Property[5] is true for spin m. Then there exists a class of data $\mathcal{P}^{(m)}$, open as a subset of $H^1_{(m)}$, such that for $u_0 \in \mathcal{P}^{(m)}$ the evolution $u(t)$ by (1.1) blows up at finite time $T_{\max}$ with the $Q^{(m)}$ profile and log-log rate. That is, for $t \in [0, T_{\max})$ there exist continuously variable parameters $\lambda(t) > 0$ and $\gamma(t) \in \mathbb{R}$ with the following properties:*

Log-log Blowup Rate::

$$\lim_{t \to T_{\max}} \|u(t)\|_{\dot{H}^1} \sqrt{\frac{T_{\max} - t}{\log|\log T_{\max} - t|}} = C \tag{1.11}$$

Description of the Singularity::

$$\lim_{t \to T_{\max}} u(t,x) - \frac{1}{\lambda(t)} Q^{(m)}\left(\frac{x}{\lambda(t)}\right) e^{-i\gamma(t)} = u^*(x) \in L^2(\mathbb{R}^2). \tag{1.12}$$

We will now discuss the consistency of the self-similar regime discovered by Fibich & Gavish and Theorem 1.1. Consider,

$$B_{\alpha,m} = \left\{ u_0 \in H^1_{(m)} : M[Q^{(m)}] < M[u_0] < M[Q^{(m)}] + \alpha \right\}.$$

Then, due to the variational characterization of $Q^{(m)}$:

Theorem 1.2 ("Orbital Stability")**.** *For $\alpha > 0$ sufficiently small, let $v \in B_{\alpha,m}$ with, $E[v] \leq \alpha \|v\|^2_{H^1}$. Then there exists $\lambda_v > 0, \gamma_v \in \mathbb{R}$ such that,*

$$\|\lambda_v \, v\,(\lambda_v y)\, e^{i\gamma_v} - Q^{(m)}(y)\|_{H^1} \leq \delta(\alpha),$$

where $\delta(\alpha) \to 0$ as $\alpha \to 0$.

The proof is by means of concentration compactness and the Gagliardo Nirenberg inequality in $H^1_{(m)}$, and is not constructive. See [28, Theorem 6] for a clear exposition. The class of data $\mathcal{P}^{(m)}$ from Theorem 1.1 belongs to $B_{\alpha,m}$, and we note that the orbital stability of Theorem 1.2 applies to all data $v_0 \in B_{\alpha,m}$ that blowup in finite time. Indeed, we conjecture[6] that finite time blowup solutions from the class $B_{\alpha,m}$ either obey the log-log blowup rate (1.11) or the lower bound, $\|v(t)\|_{H^1} \gtrsim (T_{max} - t)^{-1}$.

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[4] In particular, they present results using $u_0 = 1.02Q^{(m)}(r, \theta)$ and spin $m = 2$. The profiles identified, denoted G_m, are truncated solutions of equation (2.18) with Cauchy boundary conditions and an implied value of b, in this case $b \approx 0.1092$. Our own truncated solutions of (2.18) are very similar. Fibich & Gavish have conveyed by personal communication corresponding discoveries for spin $m = 1$ and data as small as $1.00001Q^{(m)}$.

[5] See Proposition 1.1, below.

[6] We expect the analysis of [27] to apply, and that the proof of Theorem 1.1 may be reformulated to apply to all $v_0 \in B_{\alpha,m}$ with $\|v(t)\|_{H^1} \in L^1\,(t \in [0, T_{max}))$, as in [22].

Collapse at the square-root rate has also been observed numerically in the case with no spin, [5]. These examples are an important area of continuing study. It is possible that the threshold α of Theorem 1.2 (and hence the applicability of Theorem 1.1) is exceedingly small.

1.4. **Spectral Propety.** In order to demonstrate the dynamic claimed in Theorem 1.1, we will attempt to parameterize the solution in terms of the symmetries and a suitable deformation of the profile $Q^{(m)}$. In order for the finite-dimensional system of parameters to capture the essential dynamics of the solution we require two things. First, that the parameter dynamics can be reliably predicted from a finite system of differential inequalities. Second, that after removing the central profile from the solution the error ϵ can be estimated in terms of those parameter dynamics.

That the parameter dynamic are stable is an essential feature of the log-log regime. Indeed, Raphaël showed [27] that the relationship between a particular ratio of parameters[7]and a fixed constant evolves according to a Riccati equation, with the log-log dynamic corresponding to the stable branch. To control the error ϵ in terms of the dynamics, we will consider the following operator, derived from the linearized energy,

$$\mathcal{H}^{(m)}(\epsilon,\epsilon) = \left\langle \mathcal{L}_1^{(m)}\epsilon_1, \epsilon_1 \right\rangle + \left\langle \mathcal{L}_2^{(m)}\epsilon_2, \epsilon_2 \right\rangle, \tag{1.13}$$

where

$$\mathcal{L}_1^{(m)} = -\Delta + 3Q^{(m)} y\cdot\nabla\overline{Q}^{(m)}, \quad \mathcal{L}_2^{(m)} = -\Delta + Q^{(m)} y\cdot\nabla\overline{Q}^{(m)}, \tag{1.14}$$

$$\epsilon_1 = e^{im\theta}\, Re\left(e^{-im\theta}\epsilon\right), \quad \epsilon_2 = e^{im\theta}\, Im\left(e^{-im\theta}\epsilon\right). \tag{1.15}$$

This decomposition, $\epsilon = \epsilon_1 + i\epsilon_2$, is powerful, as it reduces the algebraic structure of the problem in H^1_m to that of the radially symmetric problem in H^1. For further discussion, see (2.32), below.

We will prove the following for $m = 1$,

Proposition 1.1 (Spectral Property). *Let $\epsilon \in H^1_{(m)}$ Then there exists a universal constant δ_m such that*

$$\begin{aligned}\mathcal{H}^{(m)}(\epsilon,\epsilon) \geq & \delta_m \left(\int |\nabla_y \epsilon|^2 + \int |\epsilon|^2 e^{-|y|}\right) \\ & - \frac{1}{\delta_m}\left(\left\langle \epsilon_1, Q^{(m)}\right\rangle^2 + \left\langle \epsilon_1, \Lambda Q^{(m)}\right\rangle^2 \right. \\ & \left. + \left\langle \epsilon_2, \Lambda Q^{(m)}\right\rangle^2 + \left\langle \epsilon_2, \Lambda^2 Q^{(m)}\right\rangle^2\right).\end{aligned} \tag{1.16}$$

In the case of L^2-critical nonlinearity, no spin, and dimension $N = 1$, Merle and Raphaël [21, Appendix A] gave an explicit proof of the Spectral Property. In the case of L^2-critical nonlinearity, no spin, and dimensions $N = 2, 3, 4, 5$, including equation (1.1) in the case $m = 0$, Fibich, Merle and Raphaël [6] have given a numerical proof that inspires our own proof of Proposition 1.1 in Section 3. Details of our numerical methods are provided in Appendix B. Code to reproduce our computations is available at `http://www.math.toronto.edu/simpson/files/vortex_dist.tgz`. As stated, the spectral property is false for $m = 2, 3$.

2. Proof of Log-log Blowup

In this section, we prove Theorem 1.1 assuming Proposition 1.1. Before decomposing the solution, we introduce almost self-similar deformations of the vortex profiles that simulate the effect of symmetries that do not belong to H^1. The standard self-similar ansatz is, $u(t,x) = \frac{1}{\sqrt{2b(T-t)}} Q_b^{(m)}\left(\frac{x}{\sqrt{2b(T-t)}}\right) e^{i\omega(t)}$, which gives the following equation for the spatial profile,

$$\Delta Q_b^{(m)} - Q_b^{(m)} + ib\Lambda Q_b^{(m)} + Q_b^{(m)}|Q_b^{(m)}|^2 = 0. \tag{2.17}$$

[7]Namely the sign of $f_- = \frac{b}{\lambda} - d_0\sqrt{E_0}$. Parameter b will be introduced in Section 2.1.

We seek solutions with spin m. Remove a quadratic phase, $e^{im\theta}P_b^{(m)}(r) = Q_b^{(m)}\, e^{ib\frac{r^2}{4}}$, and assume the radial profile $P_b^{(m)}(r)$ is real valued. We seek solutions to,

$$\begin{cases} \Delta P_b^{(m)} - \left(1 + \frac{m^2}{r^2} - \frac{b^2}{4}r^2\right) P_b^{(m)} + \left(P_b^{(m)}\right)^3 = 0, \\ \lim_{r\to 0^+} r^{-m} P_b^{(m)}(r) \neq 0, \quad \lim_{r\to 0^+} \partial_r \left(r^{-m} P_b^{(m)}(r)\right) = 0. \end{cases} \tag{2.18}$$

As pointed out by Fibich and Gavish [4, Lemma 8], equation (2.18) does not admit solutions in either L^2 or $\dot{H}^1$, due to oscillations of amplitude r^{-1} outside the domain of uniform ellipticity of the linear part. The argument is due to Johnson and Pan [10]. We truncate a solution of (2.18) at an arbitrary point, chosen to allow close approximation to the vortex profile $Q^{(m)}$. Define,

$$R_b = \frac{1}{|b|}\sqrt{2 + 2\sqrt{1 + b^2 m^2}} \geq \frac{2}{|b|}. \tag{2.19}$$

Proposition 2.1 (Localized Self-Similar Profiles)**.** *Let $a > C\eta > 0$ where $C > 0$ is a fixed constant and a, η are sufficiently small parameters. Then for $|b| > 0$ sufficiently small, there exists $\widetilde{Q}_b^{(m)} \in H^1(\mathbb{R}^2)$, supported on $|y| < (1-\eta)R_b$, with the following properties.*

- *Simple Profile:*

$$\widetilde{Q}_b^{(m)} = e^{im\theta} e^{-ib\frac{|y|^2}{4}} \widetilde{P}_b^{(m)}(|y|), \quad \text{for } \widetilde{P}_b^{(m)} \text{ real-valued, non-negative.} \tag{2.20}$$

- *Algebraic Proximity to $Q^{(m)}$:*

$$\Delta \widetilde{Q}_b^{(m)} - \widetilde{Q}_b^{(m)} + ib\Lambda \widetilde{Q}_b^{(m)} + \widetilde{Q}_b^{(m)} |\widetilde{Q}_b^{(m)}|^2 = -\Psi_b, \tag{2.21}$$

  for an error term Ψ_b, supported on $(1-\eta)^2 R_b < |y| < (1-\eta)R_b$, that satisfies the estimate, $\|P(y)\nabla^k \Psi_b\|_{L^\infty} \leq e^{-\frac{C(P)}{|b|}}$, for $k = 0, 1$ and any polynomial P.

- *Uniform Proximity to $Q^{(m)}$:*

$$\|e^{C|y|}\left(\widetilde{Q}_b^{(m)} - Q^{(m)}\right)\|_{C^3} + \|e^{C|y|}\left(\frac{\partial}{\partial b}\widetilde{Q}_b^{(m)} + i\frac{|y|^2}{4}Q^{(m)}\right)\|_{C^2} \longrightarrow 0 \quad \text{as} \quad b \to 0. \tag{2.22}$$

- *Supercritical Mass and Degenerate Energy:*

$$\frac{\partial}{\partial(b^2)} \|\widetilde{Q}_b^{(m)}\|_{L^2}^2 \bigg|_{b^2=0} = \frac{1}{4}\int |x|^2 |Q^{(m)}|^2, \quad \text{denoted} \quad d_m, \quad \text{and,} \tag{2.23}$$

$$|E\left[\widetilde{Q}_b^{(m)}\right]| \leq e^{-(1+C\eta)(1-a)\frac{\pi}{|b|}}. \tag{2.24}$$

The proof of Proposition 2.1 is similar to that given by Merle and Raphaël [18,19,22] in the case of $m = 0$. An overview of the proof, and description of the particular adaptations for $m \neq 0$, is given in Appendix A.

Later in the argument, Section 2.3, we will introduce the linear radiation induced by the truncation error Ψ_b. A quantity Γ_b, related to the decay of this radiation, will be an important dynamical quantity, measuring the rate of mass ejection from the singular region. At the time we formally define Γ_b, Proposition 2.3, we will also prove the following estimate,

$$e^{-(1+C\eta)\frac{\pi}{b}} \lesssim \Gamma_b \lesssim e^{-(1-C\eta)\frac{\pi}{b}}. \tag{2.25}$$

2.1. **Decomposition & Modulation.**

Lemma 2.1 (Modulation Near $Q^{(m)}$)**.** *Suppose that $v \in H^1_{(m)}$ is close to $Q^{(m)}$, up to symmetries:*

$$v(x) = \frac{1}{\lambda_v}\left(\widetilde{Q}_{b_v}^{(m)} + \epsilon_v\right)\left(\frac{x}{\lambda}\right) e^{-i\gamma_v}, \tag{2.26}$$

for some symmetry parameters $\lambda_v > 0$, $b_v > 0$ and $\gamma_v \in \mathbb{R}$ such that the error is comparably small,

$$\int |\nabla_y \epsilon_v(y)|^2\, dy + \int_{|y| \leq \frac{10}{b_v}} |\epsilon_v|^2 e^{-|y|}\, dy < \Gamma_{b_v}^{\frac{1}{2}}, \tag{2.27}$$

where y denotes $\frac{x}{\lambda_v}$, and such that the deformed profile is sufficiently close to $Q^{(m)}$,

$$\lambda_v < \frac{1}{10} b_v \quad \text{and,} \quad b_v < \alpha^*. \tag{2.28}$$

Then there are parameters $\lambda_0 > 0$, $b_0 > 0$ and $\gamma_0 \in \mathbb{R}$, nearby in the sense,

$$|b_0 - b_v| + |\frac{\lambda_0}{\lambda_v} - 1| \leq \Gamma_{b_0}^{\frac{1}{5}}, \tag{2.29}$$

and such that the error ϵ_0 corresponding to these parameters,

$$\epsilon_0(y) = \lambda_0 \, v\,(\lambda_0 y)\; e^{i\gamma_0} - \widetilde{Q}_{b_0}^{(m)}, \tag{2.30}$$

satisfies the following orthogonality conditions[8]*:*

$$\mathrm{Re}\left\langle \epsilon_0, |y|^2 \widetilde{Q}_{b_0}^{(m)} \right\rangle = \mathrm{Im}\left\langle \epsilon_0, \Lambda^2 \widetilde{Q}_{b_0}^{(m)} \right\rangle = \mathrm{Im}\left\langle \epsilon_0, \Lambda \widetilde{Q}_{b_0}^{(m)} \right\rangle = 0. \tag{2.31}$$

Let us reiterate and extend the notation alluded to by equation (1.13),

$$\begin{aligned} &\left.\begin{aligned} \epsilon_1 &= e^{im\theta} \mathrm{Re}\left(e^{-im\theta}\epsilon\right) \\ \epsilon_2 &= e^{im\theta} \mathrm{Im}\left(e^{-im\theta}\epsilon\right) \end{aligned}\right\} \quad \Longrightarrow \epsilon = \epsilon_1 + i\epsilon_2, \\ &\left.\begin{aligned} \Sigma &= e^{im\theta} \mathrm{Re}\left(e^{-ib\frac{|y|^2}{4}} \widetilde{P}_b^{(m)}\right) \\ \Theta &= e^{im\theta} \mathrm{Im}\left(e^{-ib\frac{|y|^2}{4}} \widetilde{P}_b^{(m)}\right) \end{aligned}\right\} \quad \Longrightarrow \widetilde{Q}_b^{(m)} = \Sigma + i\Theta. \end{aligned} \tag{2.32}$$

Products between the components of ϵ and $\widetilde{Q}_b^{(m)}$ behave as if they were real-valued, as does the modulus, for example, $|\epsilon|^2 = |\epsilon_1|^2 + |\epsilon_2|^2$. Moreover, since $|y|^2$ and the scaling operator, $\Lambda = 1 + y \cdot \nabla_y$, are radial operators, the algebraic relations for $|y|^2 \widetilde{Q}_b^{(m)}$ and $\Lambda \widetilde{Q}_b^{(m)}$ are exactly the same as the case $m = 0$, [19, Proposition 9 (iii)]. In particular, one may verify that, $\mathcal{L}_1^{(m)}(\epsilon_1) = \frac{1}{2}\left[L_+^{(m)}(\Lambda\epsilon_1) - \Lambda(L_+^{(m)}\epsilon_1)\right]$, is true regardless of m. This is the essential relationship for Lemma 2.2, below. In the notation of (2.32), the orthogonality conditions of equation (2.31) can be written,

$$\begin{aligned} &\langle \epsilon_1, |y|^2 \Sigma \rangle + \langle \epsilon_2, |y|^2 \Theta \rangle = 0, \\ &\langle \epsilon_2, \Lambda^2 \Sigma \rangle - \langle \epsilon_1, \Lambda^2 \Theta \rangle = 0, \\ &\langle \epsilon_2, \Lambda \Sigma \rangle - \langle \epsilon_1, \Lambda \Theta \rangle = 0. \end{aligned}$$

These are exactly the same form as in the case $m = 0$. Indeed, the proof of Lemma 2.1, an implicit function argument, is identical. See [28, Lemma 2] for a clear exposition. For $m = 0$, the following Lemma was proven by Merle and Raphaël [19, equation (116)], and the same proof applies here.

Lemma 2.2. *Let $\epsilon \in H^1_{(m)}$, and assume the Spectral Property is true. Then,*

$$\begin{aligned} &\left\langle L_1^{(m)} \epsilon_1, \epsilon_1 \right\rangle - \frac{\left\langle \epsilon_1, L_+^{(m)} \Lambda^2 Q^{(m)} \right\rangle \left\langle \epsilon_1, \Lambda Q^{(m)} \right\rangle}{\|\Lambda Q^{(m)}\|_{L^2}^2} \geq \\ &\qquad \delta_m \left(\int |\nabla_y \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) - \frac{1}{\delta_m} \left(\left\langle \epsilon_1, Q^{(m)} \right\rangle^2 + \left\langle \epsilon_1, |y|^2 Q^{(m)} \right\rangle^2 \right), \end{aligned} \tag{2.33}$$

Definition 2.2 (Description of Initial Data)**.** *Define $\mathcal{P}^{(m)}$ to be those functions $u_0 \in H^1_{(m)}$ for which there are parameters $\lambda_0 > 0$, $b_0 > 0$ and $\gamma_0 \in \mathbb{R}$ that satisfy the following conditions. Let ϵ_0 denote the error in approximating u_0 with these particular parameters,*

$$u_0(x) = \frac{1}{\lambda_0} \left(\widetilde{Q}_{b_0}^{(m)} + \epsilon_0 \right) \left(\frac{x}{\lambda_0} \right) e^{-i\gamma_0}. \tag{2.34}$$

We require that the orthogonality conditions (2.31) are satisfied, that there is,

[8] These orthogonality conditions were introduced [19, Lemma 6], and lead to a better estimate on the phase parameter than achieved in [18].

proximity to $Q^{(m)}$,:

$$
\begin{aligned}
&\textit{in } L^2: \quad 0 < b_0^2 + \|\epsilon\|_{L^2}^2 < (\alpha^*)^2, \\
&\textit{in } \dot{H}^1: \quad \int |\nabla_y \epsilon_0(y)|^2\, dy + \int_{|y| \le \frac{10}{b_0}} |\epsilon_0(y)|^2 e^{-|y|}\, dy < \Gamma_{b_0}^{\frac{6}{7}},
\end{aligned}
\tag{2.35}
$$

parameters consistent with the log-log rate,:

$$
e^{-e^{\frac{2\pi}{b_0}}} < \lambda_0 < e^{-e^{\frac{\pi}{2}\frac{1}{b_0}}}, \quad \textit{and,}
\tag{2.36}
$$

normalized energy,:

$$
\lambda_0^2 |E_0| < \Gamma_{b_0}^{10}.
\tag{2.37}
$$

Remark 2.3 ($\mathcal{P}^{(m)}$ is Non-Empty). Choose b_0 and λ_0 to satisfy (2.35) and (2.36). Let $f \in H^1_{(m)}$ satisfy orthogonality conditions (2.31) with $\|f\|_{H^1} = 1$, $\langle f, Q^{(m)} \rangle = 1$. Such an f may be computed explicitly from $Q^{(m)}$. Note that $\partial_\nu E[Q^{(m)} + \nu f]\big|_{\nu=0} = -\langle F, Q^{(m)} \rangle = -1$, and therefore we may choose $\epsilon_0 = \nu f$ with ν of the order of $E[Q_b^{(m)}]$ to satisfy (2.24).

For the remainder of this section, we consider a fixed representative $u_0 \in \mathcal{P}^{(m)}$. By the continuity of the flow of (1.1) in H^1, and Lemma 2.1, there exists continuous functions $\lambda(t) > 0$, $b(t) > 0$ and $\gamma(t) \in \mathbb{R}$ and some maximal $T_{\text{hyp}} \in (0, T_{\max}]$ such that the following relaxations of (2.35), (2.36) and (2.37) hold for all $t \in [0, T_{\text{hyp}})$:

$$
0 < b^2(t) + \|\epsilon(t)\|_{L^2}^2 < (\alpha^*)^{\frac{1}{5}},
\tag{2.38}
$$

$$
\int |\nabla_y \epsilon(t,y)|^2\, dy + \int_{|y| \le \frac{10}{b(t)}} |\epsilon(t,y)|^2 e^{-|y|}\, dy < \Gamma_{b(t)}^{\frac{3}{4}},
\tag{2.39}
$$

$$
e^{-e^{\frac{10\pi}{b(t)}}} < \lambda(t) < e^{-e^{\frac{\pi}{10}\frac{1}{b(t)}}}, \quad \text{and,}
\tag{2.40}
$$

$$
\lambda^2(t) |E_0| < \Gamma_{b(t)}^2.
\tag{2.41}
$$

Note that as a consequence of these hypotheses, we may apply Lemma 2.1 at any $t \in [0, T_{\text{hyp}})$. Therefore, one of the following occurs:

Case 1:: $T_{\text{hyp}} < T_{\max}$, and one of the hypotheses fails at $t = T_{\text{hyp}}$, or
Case 2:: $T_{\text{hyp}} = T_{\max}$, $b \to 0$ as $t \to T_{\max}$, and due to (2.40) we have blowup.

In this section we will show that **Case 1** cannot occur. Then, assuming **Case 2**, we will derive the conclusions of Theorem 1.1.

Remark 2.4 (Parameters). The parameter $\eta > 0$, already introduced, relates to the cutoff and shape of the singular profile $\widetilde{Q}_b^{(m)}$. Parameter $a > 0$, to be introduced in Section 2.3, will be related to a cutoff point of the linear radiation associated with $\widetilde{Q}_b^{(m)}$. The value of η is determined by the value of a so that the argument of Subsection 2.4.2 is successful. At all times, $\alpha^* > 0$ is assumed sufficiently small for all the appropriate constants to cooperate.

2.2. Conservation Laws & Basic Estimates.

By substitution of the time-dependent version of the geometric decomposition (2.34), the conservation laws of (1.1) and the orthogonality conditions (2.31) lead to some basic estimates.

Lemma 2.5. *For all $t \in [0, T_{\text{hyp}})$,*

due to conservation of mass,:

$$
b^2 + \int |\epsilon|^2 \lesssim (\alpha^*)^{\frac{1}{2}},
\tag{2.42}
$$

due to conservation of energy,:

$$
\begin{aligned}
&2\mathrm{Re}\left\langle \epsilon, \widetilde{Q}_b^{(m)} - ib\Lambda\widetilde{Q}_b^{(m)} - \Psi_b \right\rangle \sim \int |\nabla_y \epsilon|^2\, dy - 3\int_{|y|\leq \frac{10}{b}} |Q^{(m)}\epsilon_1|^2 - \int_{|y|\leq \frac{10}{b}} |Q^{(m)}\epsilon_2|^2, \\
&\qquad \textit{with error of the order,} \quad \Gamma_b^{1-C\eta} + \delta(\alpha^*)\left(\int |\nabla_y \epsilon|^2\, dy + \int_{|y|\leq \frac{10}{b}} |\epsilon|^2 e^{-|y|}\, dy\right).
\end{aligned} \tag{2.43}
$$

Proof. To prove (2.42), expand the conservation of mass,

$$
\int |\widetilde{Q}_b^{(m)}|^2\, dy - \int |Q^{(m)}|^2\, dy + 2Re\left\langle \epsilon, \widetilde{Q}_b^{(m)}\right\rangle + \int |\epsilon|^2 = \int |u_0|^2 - \int |Q^{(m)}|^2\, dy.
$$

Recognize $\partial_{(b^2)}\|\widetilde{Q}_b^{(m)}\|_{L^2}^2$, and recall (2.23). Use initial condition (2.35) and hypothesis (2.39). To prove (2.43), expand the conservation of energy, as in [19, eqn (188)]. Use the normalized energy (2.41) to estimate $\lambda^2 E_0$. For the terms $\mathcal{O}(\epsilon^3)$, use the exponential decay of $Q^{(m)}$, the Hardy-type inequalities below, and hypothesis (2.39). $\square$

Lemma 2.6 (Hardy-type Inequalities)**.** *For any $\kappa > 0$ and for all $v \in H^1(\mathbb{R}^2)$,*

$$
\int_{y\in\mathbb{R}^2} |v(y)|^2 e^{-\kappa|y|} \leq C(\kappa)\left(\int |\nabla v(y)|^2 + \int_{|y|\leq 1} |v(y)|^2 e^{-|y|}\right), \tag{2.44}
$$

$$
\int_{|y|\leq\kappa} |v(y)|^2 \leq C\,\kappa^2 \log\kappa \left(\int |\nabla v(y)|^2 + \int_{|y|\leq 1} |v(y)|^2 e^{-|y|}\right). \tag{2.45}
$$

Proof. Equation (2.45) is proven [22, equation (4.11)], and the same techniques prove (2.44). $\square$

Let us reiterate the notation $y = \frac{x}{\lambda(t)} \in \mathbb{R}^2$, and introduce a rescaled time,

$$
s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} + s_0 \quad \text{where} \quad s_0 = e^{\frac{3\pi}{4b_0}} \quad \text{and,} \quad s_1 = s(T_{\mathrm{hyp}}) \in (s_0, \infty]. \tag{2.46}
$$

In these new variables, equation (1.1) now reads,

$$
\begin{aligned}
ib_s\frac{\partial}{\partial b}\widetilde{Q}_b^{(m)} + i\epsilon_s - M(\epsilon) + ib\Lambda\epsilon =& i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\widetilde{Q}_b^{(m)} + \tilde{\gamma}_s\widetilde{Q}_b^{(m)} \\
&+ i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\epsilon + \tilde{\gamma}_s\epsilon \\
&+ \Psi_b - R[\epsilon],
\end{aligned} \tag{2.47}
$$

where we introduced the new variable, $\tilde{\gamma}(s) = -s-\gamma(s)$, the term $R[\epsilon]$ corresponds to those terms of $u|u|^2$ that are formally $\mathcal{O}(\epsilon^2)$, and M is the linearized operator near $\widetilde{Q}_b^{(m)}$, analogous to $L^{(m)}$, (1.10). Using our choice of notation (2.32), equation (2.47) has exactly the same form as that given by Merle & Raphaël [19, Lemma 7] in the case of $m = 0$. Indeed, the algebraic structure in $H^1_{(m)}$ is the same, and the arguments of [19, Appendix C] (or [27, Appendix A]) prove the following Lemma, without modification.

Lemma 2.7. *For all $s \in [s_0, s_1)$,*

due to orthogonality with $|y|^2\widetilde{Q}_b^{(m)}$, $\Lambda\widetilde{Q}_b^{(m)}$, and estimate (2.43),:

$$
\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| \lesssim \Gamma_b^{1-C\eta} + \left(\int |\nabla_y \epsilon|^2\, dy + \int_{|y|\leq \frac{10}{b}} |\epsilon|^2 e^{-|y|}\, dy\right), \tag{2.48}
$$

due to orthogonality with $\Lambda^2\widetilde{Q}_b^{(m)}$,:

$$
\left|\tilde{\gamma}_s - \frac{\left\langle \epsilon_1, L_+^{(m)}\Lambda^2 Q^{(m)}\right\rangle}{\|\Lambda Q^{(m)}\|_{L^2}^2}\right| \leq \Gamma_b^{1-C\eta} + \delta(\alpha^*)\left(\int |\nabla_y \epsilon|^2\, dy + \int_{|y|\leq \frac{10}{b}} |\epsilon|^2 e^{-|y|}\, dy\right)^{\frac{1}{2}}. \tag{2.49}
$$

In order to show the coercive control (2.39) does not fail, we prove the following Local Virial Identity. This estimate was originally shown by Merle & Raphaël in [21] and was inspired by the work of Martel & Merle [13] in a proof of soliton stability for the generalized Korteweg-de Vries equation.

Lemma 2.8 (Local Virial Identity)**.** *For all $s \in [s_0, s_1)$,*

$$b_s \geq \delta_1 \left(\int |\nabla_y \epsilon|^2 \, dy + \int_{|y| \leq \frac{10}{b}} |\epsilon|^2 e^{-|y|} \, dy \right) - \Gamma_b^{1-C\eta}, \tag{2.50}$$

where $\delta_1 > 0$ is a universal constant.

Proof Outline. We begin the same as the proof of (2.48): take the real part of the inner product of (2.47) by $\Lambda \widetilde{Q}_b^{(m)}$ and use (2.43) to eliminate the terms $\mathcal{O}(\epsilon)$, as in [19, Appendix C]. The interim result is,

$$\begin{aligned} b_s \frac{1}{4} \|y Q^{(m)}\|_{L^2}^2 &\gtrsim \mathcal{H}^{(m)}(\epsilon, \epsilon) - \tilde{\gamma}_s \left(\epsilon_1, \Lambda Q^{(m)} \right) \\ &\quad - \Gamma_b^{1-C\eta} - \delta(\alpha^*) \left(\int |\nabla_y \epsilon|^2 \, dy + \int_{|y| \leq \frac{10}{b}} |\epsilon|^2 e^{-|y|} \, dy \right), \end{aligned} \tag{2.51}$$

where we have used the preliminary estimate (2.48). We have also used the proximity of $\widetilde{Q}_b^{(m)}$ to $Q^{(m)}$, (2.22), to isolate the b-dependence from interactions of the form $\left(\widetilde{Q}_b^{(m)} \right)^2 \epsilon^2$ as a lower-order potential, the same form as [19, equation (215)], here included as part of the final term. To prove the Local Virial Identity, use the preliminary estimate on $\tilde{\gamma}_s$ and the Spectral Property, Proposition 1.1, as adapted by Lemma 2.2. $\square$

2.3. **Lyapounov Functional.** We cannot hope to prove b^2 is monotonically decreasing, since it is a modulation parameter, and thus cannot hope to control ϵ by the local virial identity at all times. In this section we prove a Lyapounov functional based on the mass ejection from the singular region, to which b^2 is related, (2.23), and which we expect b^2 to track. To do this, we will further approximate the central profile by including a linear radiative tail.

Proposition 2.3 (Linear Radiation)**.** *For $\eta > 0$ sufficiently small, and all $|b| > 0$ sufficiently small depending on η, there exists a unique solution $\zeta_b^{(m)} \in \dot{H}^1_{(m)}(\mathbb{R}^2)$ to*

$$\Delta \zeta_b^{(m)} - \zeta_b^{(m)} + ib\Lambda \zeta_b^{(m)} = \Psi_b, \tag{2.52}$$

where Ψ_b is the truncation error given by (2.21). Radiation $\zeta_b^{(m)} \notin L^2(\mathbb{R}^2)$, and, moreover, $\lim_{|y| \to +\infty} |y| |\zeta_b^{(m)}(y)|^2$ exists. We denote this decay rate as, Γ_b.

- *Size in $\dot{H}^1$ and Derivative by b:*

$$\|\zeta_b^{(m)}\|_{\dot{H}^1}^2 \leq \Gamma_b^{1-C\eta}, \quad \text{and,} \quad \|\frac{\partial}{\partial b} \zeta_b^{(m)}\|_{C^1} \leq \Gamma_b^{\frac{1}{2} - C\eta}. \tag{2.53}$$

- *Decay past the support of Ψ_b:*

$$\| |y| |\zeta_b^{(m)}| + |y|^2 |\nabla \zeta_b^{(m)}| \|_{L^\infty(|y| \geq R_b)} \leq \Gamma_b^{\frac{1}{2} - C\eta} < +\infty. \tag{2.54}$$

- *Stronger decay far past the support of Ψ_b:*

$$e^{-(1+C\eta)\frac{\pi}{b}} \leq \frac{4}{5} \Gamma_b \leq \| |y|^2 |\zeta_b^{(m)}|^2 \|_{L^\infty(|y| \geq R_b^2)} \leq e^{-(1-C\eta)\frac{\pi}{b}},$$

  which we have already discussed, equation (2.25), and,

$$\| |y|^2 |\nabla \zeta_b^{(m)}| \|_{L^\infty(|y| \geq R_b^2)} \leq C \frac{\Gamma_b^{\frac{1}{2}}}{|b|}. \tag{2.55}$$

The proof of Proposition 2.3 is given due to Merle and Raphaël [19, Appendix E] and [22, Appendix A]. Brief discussion of the necessary adapatations will be given at the end of Appendix A.

We denote,

$$A(t) = e^{a \frac{\pi}{b(t)}}, \quad \text{so that,} \quad \Gamma_b^{-\frac{a}{2}} \leq A \leq \Gamma_b^{-\frac{3a}{2}}, \tag{2.56}$$

where $a > 0$ is a universal parameter. Let ϕ_A denote a smooth cutoff function of the region, $\mathbb{1}_{\{|y|>2A\}}$. The truncated radiation, $\widetilde{\zeta}_b^{(m)} = (1-\phi_A)\zeta_b^{(m)}$, is algebraically close to $\zeta_b^{(m)}$ and satisfies,

$$\Delta\widetilde{\zeta}_b^{(m)} - \widetilde{\zeta}_b^{(m)} + ib\Lambda\widetilde{\zeta}_b^{(m)} = \Psi_b + F, \quad \text{where,} \quad |F|_{L^\infty} + |y\cdot\nabla F|_{L^\infty} \lesssim \frac{\Gamma_b^{\frac{1}{2}}}{A}. \tag{2.57}$$

We will now repeat the calculation of the local virial identity, this time including the linear radiation $\widetilde{\zeta}_b$ as part of the central profile. That is we write,

$$\widetilde{\epsilon} = \epsilon - \widetilde{\zeta}_b^{(m)} \quad \Rightarrow \quad u(t,x) = \frac{1}{\lambda(t)}\left(\widetilde{Q}_b^{(m)} + \widetilde{\zeta}_b^{(m)} + \widetilde{\epsilon}\right)\left(\frac{x}{\lambda}\right)e^{-i\gamma(t)}, \tag{2.58}$$

without affecting the parameters. This leads to a refined version of equation (2.47) for $\widetilde{\epsilon}$. The proof of the following three Lemmas is virtually identical[9]to that of Merle and Raphaël, [22, Chapter 4].

Lemma 2.9 (Radiative Virial Identity)**.** *For all $s \in [s_0, s_1)$,*

$$\partial_s f_1 \geq \delta_2\left(\int |\nabla_y\widetilde{\epsilon}|^2\,dy + \int_{|y|\leq\frac{10}{b}} |\widetilde{\epsilon}|^2 e^{-|y|}\,dy\right) + \Gamma_b - \frac{1}{\delta_2}\int_{A\leq|y|\leq 2A}|\epsilon|^2\,dy, \tag{2.59}$$

where $\delta_2 > 0$ is a universal constants and,

$$f_1(s) = \frac{b}{4}\|y\widetilde{Q}_b^{(m)}\|_{L^2}^2 + \frac{1}{2}\mathrm{Im}\left(\int y\cdot\nabla\widetilde{Q}_b^{(m)}\overline{\widetilde{\zeta}}_b^{(m)}\right) + \mathrm{Im}\left\langle\epsilon, \Lambda\widetilde{\zeta}_b^{(m)}\right\rangle.$$

In the light of estimates such as (2.45) we cannot expect the radiative virial identity to give a good control for ϵ. Let ϕ_∞ denote a smooth cutoff function of the region $\mathbb{1}_{\{|y|>3A\}}$ with steady derivative $\phi_\infty' \approx \frac{1}{3A}$ on the region $A \leq |y| \leq 2A$.

Lemma 2.10 (Mass-Ejection)**.**

$$\partial_s\left(\int\phi_\infty\left(\frac{y}{A}\right)|\epsilon|^2\,dy\right) \geq \frac{b}{400}\int_{A\leq|y|\leq 2A}|\epsilon|^2\,dy - \Gamma_b^{\frac{a}{2}}\int|\nabla_y\epsilon|^2\,dy - \Gamma_b^2. \tag{2.60}$$

Remark 2.11. As a heuristic, assume that $\epsilon \approx \zeta_b$ on the region, $A \leq |y| \leq 2A$. Use the definition of Γ_b to approximate the mass. Then with hypothesis (2.39), Lemma 2.10 suggests continuous ejection of mass from the region $|y| < \frac{A}{2}$, regardless of whether that region is growing or contracting.

Together with the conservation of mass, Lemma 2.9 and Lemma 2.10 prove the following Lemma. The argument relies on (2.53) and the relation between parameters a and η stipulated by Proposition 2.1.

Lemma 2.12 (Lyapounov Functional)**.** *For all $s \in [s_0, s_1)$,*

$$\partial_s\mathcal{J} \leq -Cb\left(\Gamma_b + \int|\nabla_y\widetilde{\epsilon}|^2\,dy + \int_{|y|\leq\frac{10}{b}}|\widetilde{\epsilon}|^2e^{-|y|}\,dy + \int_{A\leq|y|\leq 2A}|\epsilon|^2\right), \tag{2.61}$$

where $C > 0$ is a universal constant,

$$\begin{aligned}\mathcal{J}(s) =& \|\widetilde{Q}_b^{(m)}\|_{L^2}^2 - \|Q^{(m)}\|_{L^2}^2 \\ &+ 2\langle\epsilon_1, \Sigma\rangle + 2\langle\epsilon_2, \Theta\rangle + \int(1-\phi_\infty)|\epsilon|^2\,dy \\ &- \frac{\delta_2}{800}\left(b\widetilde{f}_1(b) - \int_0^b\widetilde{f}_1(v)\,dv + b\,Im\left(\epsilon, \Lambda\widetilde{\zeta}_b^{(m)}\right)\right),\end{aligned} \tag{2.62}$$

and $\widetilde{f}_1$ is the principal part of f_1,

$$\widetilde{f}_1(b) = \frac{b}{4}\|y\widetilde{Q}_b^{(m)}\|_{L^2}^2 + \frac{1}{2}\mathrm{Im}\left(\int y\cdot\nabla\widetilde{\zeta}_b^{(m)}\overline{\widetilde{\zeta}}_b^{(m)}\right).$$

[9]Where Merle and Raphaël write, $\widetilde{\zeta}_b = \widetilde{\zeta}_{re} + i\widetilde{\zeta}_{im}$, one should instead read, $\widetilde{\zeta}_b^{(m)} = \widetilde{\zeta}_1 + i\widetilde{\zeta}_2$, each component with spin m following the convention of equation (2.32).

2.3.1. *Estimates on $\mathcal{J}$.* To first order, $\mathcal{J}$ quantifies the excess mass remaining in the singular region. After explicitly accounting for this mass, $\mathcal{J}$ is comparable to $\|\epsilon\|_{H^1}^2$, up to a power of Γ_b that depends on our choice of truncation of the radiation.

Lemma 2.13. *For all $s \in [s_0, s_1)$ we have the crude estimate,*

$$|\mathcal{J} - d_m b^2| < \delta_3 b^2, \tag{2.63}$$

where $0 < \delta_3 \ll 1$ is a universal constant, and $d_m b^2$ is the approximate excess mass of profile $\widetilde{Q}_b^{(m)}$.

Lemma 2.14. *Let f_2 denote those terms of $\mathcal{J}$ that are formally $\mathcal{O}(b^2)$,*

$$f_2(b) = \|\widetilde{Q}_b^{(m)}\|_{L^2}^2 - \|Q^{(m)}\|_{L^2}^2 - \frac{\delta_2}{800}\left(b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)\,dv\right).$$

These are the terms concerned with the excess mass. For all $s \in [s_0, s_1)$ we have the refined estimate,

$$\mathcal{J}(s) - f_2(b(s)) \begin{cases} \leq \Gamma_b^{1-Ca} & +CA^2\log A\left(\int |\nabla_y \epsilon|^2\,dy + \int_{|y|\leq \frac{10}{b}} |\epsilon|^2 e^{-|y|}\,dy\right) \\ \geq -\Gamma_b^{1-Ca} & +\frac{1}{C}\left(\int |\nabla_y \epsilon|^2\,dy + \int_{|y|\leq \frac{10}{b}} |\epsilon|^2 e^{-|y|}\,dy\right). \end{cases} \tag{2.64}$$

Proof. The crude estimate (2.63) can be either proven directly or seen as a special case of (2.64) and hypothesis (2.39). The estimate (2.64) and its proof is exactly as given by Merle & Raphaël, [22, equation (5.6)]. The essential point is that the most difficult term of $\mathcal{J}(s) - f_2(b(s))$ can be handled with the conservation of energy (1.3), here written in rescaled variables,

$$\begin{aligned} 2\langle \epsilon_1, \Sigma\rangle + 2\langle \epsilon_2, \Theta\rangle + \int (1-\phi_A)|\epsilon|^2 = \left\langle L_+^{(m)}\epsilon_1, \epsilon_1\right\rangle + \left\langle L_-^{(m)}\epsilon_2, \epsilon_2\right\rangle - \int \phi_A|\epsilon|^2 \\ + E[\widetilde{Q}_b^{(m)}] - \lambda^2 E_0 + \mathcal{O}(\epsilon^3). \end{aligned}$$

To establish the lower bound of (2.64) we need $L^{(m)}$ to be coercive. We claim that,

Lemma 2.15. *For $v = v_1 + iv_2 \in H^1_{(m)}$,*

$$\left\langle L_+^{(m)} v_1, v_1\right\rangle + \left\langle L_-^{(m)} v_2, v_2\right\rangle \geq \delta_3\|v\|_{H^1}^2 - \frac{1}{\delta_3}\left(\langle v_1, \phi_+\rangle^2 + \left\langle v_1, \nabla Q^{(m)}\right\rangle^2 + \left\langle v_2, Q^{(m)}\right\rangle^2\right), \tag{2.65}$$

where ϕ_+ is the normalized eigenvector corresponding to the smallest eigenvalue of $L_+^{(m)}$.

Merle and Raphaël [22, Appendix D] remark that ϕ_+ lies in the span of $Q^{(m)}$ and $|y|^2 Q^{(m)}$, and that (2.65) may be localized to,

$$\begin{aligned} &\left\langle L_+^{(m)} v_1, v_1\right\rangle + \left\langle L_-^{(m)} v_2, v_2\right\rangle - \int \phi_A |v|^2 \\ &\qquad \geq \delta_2\|v\|_{H^1}^2 - \frac{1}{\delta_2}\left(\left\langle v_1, Q^{(m)}\right\rangle^2 + \left\langle v_1, |y|^2 Q^{(m)}\right\rangle^2 + \left\langle v_2, Q^{(m)}\right\rangle^2\right), \end{aligned}$$

assuming A is sufficiently large relative to the exponential decay of $Q^{(m)}$. □

Proof of Lemma 2.15. Following the variational characterization of $Q^{(m)}$, Weinstein [32, Prop 2.7] argues (in the case $m = 0$) that for all $f \in H^1_{(m)}$, $\partial_\epsilon^2 J[Q^{(m)} + \epsilon f]\big|_{\epsilon=0} \geq 0$. By an explicit calculation we conclude,

$$\inf_{f \in H^1_{(m)}, \langle f, Q^{(m)}\rangle = 0} \left\langle L_+^{(m)} f, f\right\rangle \geq 0.$$

Let $\mu_+ < 0$ be the lowest eigenvalue of $L_+^{(m)}$, and $\phi_+ \in L^2$ the corresponding normalized eigenvector. If there were two linearizely independent negative directions, then there would be one perpendicular to $R^{(m)}$. Therefore,

$$\inf_{f \in H^1_{(m)}, \langle f, \phi_+\rangle = 0} \left\langle L_+^{(m)} f, f\right\rangle \geq 0.$$

The following argument due to [14] is an improvement on the proof of [32, Prop 2.9]. Consider,

$$\delta_+ = \inf \left\{ \left\langle L_+^{(m)} f, f \right\rangle \quad | \quad \|f\|_{H^1_{(m)}} = 1 \quad \text{and} \quad \langle f, \phi_+ \rangle = 0 \right\} \geq 0.$$

Assume $\delta_+ = 0$. Then by weak convergence of a minimizing sequence there exists a minimizer $f_+ \in H^1_{(m)}$, and there are lagrange multipliers so that,

$$\left(L_+^{(m)} - \ell_1 \right) f_+ = \ell_2 \phi_+.$$

An inner product with f_+ implies $\ell_1 = 0$, and then an inner product by ϕ_+ implies $\ell_2 = 0$. As we remarked in Section 1.2, Pego & Warchall [26] found that the nullspace of $L_+^{(m)}$ restricted to $H^1_{(m)}$ is empty, and we have a contradiction. □

2.4. **Description of the Blowup.** Let us consider the hypotheses of Section 2.1 in turn. In each case, we will show that if the solution exists for $t = T_{hyp}$, then the hypothesis holds for some interval $[0, T_{\text{hyp}} + \delta)$. This will prove that **Case 1**, introduced in Section 2.1, cannot occur, and that, therefore, $T_{\text{hyp}} = T_{\max}$. This means that the dynamics described by (2.38), (2.39), (2.40) and (2.41) persist for the remaining lifetime of the solution. Indeed, we will show that that lifetime is finite, equation (2.18), and use these dynamics to prove the behaviour claimed for Theorem 1.1.

2.4.1. *Hypothesis (2.38).* Preliminary estimate (2.42) shows that hypothesis (2.38) cannot fail.

2.4.2. *Hypothesis (2.39).*

Lemma 2.16. *For all $s \in [s_0, s_1)$,*

$$\int |\nabla_y \epsilon|^2 \, dy + \int_{|y| \leq \frac{10}{b}} |\epsilon|^2 e^{-|y|} \, dy \leq \Gamma_b^{\frac{4}{5}}, \tag{2.66}$$

which shows that hypothesis (2.39) cannot fail.

Proof. Consider arbitrary fixed $s \in [s_0, s_1)$.

(a) If $\partial_s b(s) \leq 0$, then (2.66) follows from the local virial identity, (2.50).

(b) If $\partial_s b(s) > 0$, then there exists a largest interval (s_+, s), with $s_0 \leq s_+$, on which $\partial_s b > 0$.

This implies, $\quad b(s_+) < b(s) \quad$ and either, $\quad$ **(c)** $s_+ = s_0$, or, **(d)** $\partial_s b(s_+) = 0$.

In case **(c)** or **(d)**, either by the initial condition (2.35) or the local virial identity, respectively,

$$\int |\nabla_y \epsilon(s_+, y)|^2 \, dy + \int_{|y| \leq \frac{10}{b(s_+)}} |\epsilon(s_+, y)|^2 e^{-|y|} \, dy \leq \Gamma_{b(s_+)}^{\frac{6}{7}}.$$

From the upper bound of refined estimate (2.64), and assuming $a > 0$ is sufficiently small,

$$\mathcal{J}(s_+) - f_2(b(s_+)) \leq \Gamma_{b(s_+)}^{\frac{5}{6}} < \Gamma_{b(s)}^{\frac{5}{6}}. \tag{2.67}$$

Since $\mathcal{J}$ is non-increasing, and from the lower bound of refined estimate (2.64),

$$\begin{aligned} \Gamma_{b(s)}^{\frac{5}{6}} \geq& \mathcal{J}(s) - f_2(b(s_+)) \\ \gtrsim& \left(\int |\nabla_y \epsilon(s, y)|^2 \, dy + \int_{|y| \leq \frac{10}{b(s)}} |\epsilon(s, y)|^2 e^{-|y|} \, dy \right) \\ &- \Gamma_{b(s)}^{1-Ca} + \left(f_2(b(s)) - f_2(b(s_+)) \right). \end{aligned} \tag{2.68}$$

As noted in the proof of crude estimate (2.63), we may assume the constant δ_2 of equation (2.59) is sufficiently small relative to d_0, such that $0 < \left. \frac{\partial f_2}{\partial_{b^2}} \right|_{b^2 = 0} < \infty$, and proving that $(f_2(b(s)) - f_2(b(s_+))) > 0$. Assuming $a > 0$ is sufficiently small, this proves (2.66).

□

2.4.3. *Hypothesis (2.40).*

Lemma 2.17. *For all* $s \in [s_0, s_1)$,

$$b(s) \geq \frac{3\pi}{4\log s}, \quad \textit{and,} \quad \lambda(s) \leq \sqrt{\lambda_0} e^{-\frac{\pi}{3}\frac{s}{\log s}}. \tag{2.69}$$

Proof. Recall the bounds on Γ_b, (2.25), hypothesis (2.39), inject into the local virial identity (2.50), carefully integrate, and recall the clever choice of s_0, (2.46),

$$\partial_s e^{+\frac{3\pi}{4b}} = -\frac{b_s}{b^2}\frac{3\pi}{4} e^{+\frac{3\pi}{4b}} \leq 1 \implies e^{+\frac{3\pi}{4b}} \leq s - s_0 + e^{+\frac{3\pi}{4b_0}} = s.$$

Next, we take hypothesis (2.39) and preliminary estimate (2.48) to approximate the dynamics of λ,

$$|\frac{\lambda_s}{\lambda} + b| + |b_s| < \Gamma_b^{\frac{1}{2}} \implies -\frac{\lambda_s}{\lambda} \geq \frac{2b}{3} \geq \frac{\pi}{2\log s}. \tag{2.70}$$

By the initial condition on b_0, (2.35), we may assume s_0 is sufficiently large so that, $\int_{s_0}^{s} \frac{\pi}{2\log\sigma} d\sigma \geq \frac{\pi}{3}\left(\frac{s}{\log s} - \frac{s_0}{\log s_0}\right)$. By the initial choice of a log-log relationship, (2.36), we may assume, $-\log\lambda_0 \geq e^{\frac{\pi}{2b_0}} = s_0^{\frac{3}{2}}$. That is, by integrating (2.70) we have,

$$-\log\lambda \geq -\frac{1}{2}\log\lambda_0 + \frac{\pi}{3}\frac{s}{\log s}.$$

□

Corollary 2.18.

$$T_{\text{hyp}} = \int_{s_0}^{s_1} \lambda^2(\sigma)\, d\sigma \leq \lambda_0 \int_2^{+\infty} e^{-\frac{2\pi}{3}\frac{\sigma}{\log\sigma}}\, d\sigma < \alpha^*.$$

Corollary 2.19.

$$\lambda \leq e^{-e^{\frac{\pi}{5b}}}, \quad \textit{which shows that half of hypothesis (2.40) cannot fail.}$$

Proof. Due to equation (2.69), and again assuming $s_0 > 0$ is sufficiently large,

$$-\log(s\lambda(s)) \geq \frac{\pi}{3}\frac{s}{\log s} - \log s \geq \frac{s}{\log s}.$$

Take the logarithm and apply equation (2.69) again,

$$\log|-\log(s\lambda(s))| \geq \log\left(\frac{s}{\log s}\right) \geq \frac{4}{15}\log s \geq \frac{\pi}{5b(s)} \implies s\lambda(s) \leq e^{-e^{\frac{\pi}{5b}}}. \tag{2.71}$$

□

Lemma 2.20. *For all* $s \in [s_0, s_1)$,

$$b(s) \leq \frac{4\pi}{3\log s}. \tag{2.72}$$

Proof of Lemma 2.20. Due to the crude estimate (2.63) and the Lyapounov inequality (2.61),

$$\partial_s e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}} \gtrsim \frac{b}{\mathcal{J}}\Gamma_b e^{\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}} \geq 1.$$

where the final inequality is due to $\frac{5}{4} > 1 + C\eta$, the bound for Γ_b, (2.25), and assumes α^* is sufficiently small. By integrating the inequality,

$$e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s)}}} \geq e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s_0)}}} + s - s_0. \tag{2.73}$$

Finally, by the crude estimate (2.63) and the definition of s_0 (2.46),

$$e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s_0)}}} > e^{\frac{\pi}{b_0}} > s_0,$$

which, again with the crude estimate (2.63), proves (2.72) from (2.73). Finally, we note here a related estimate that will be used in Subsection 2.4.5. Divide the Lyapounov inequality (2.61) by $\sqrt{\mathcal{J}}$, integrate in time, and use the crude estimate once again to get,

$$\int_{s_0}^{s}\left(\Gamma_{b(\sigma)}+\int|\nabla_y\epsilon|^2\,dy+\int_{|y|\le\frac{10}{b}}|\epsilon|^2e^{-|y|}\,dy\right)d\sigma\lesssim\sqrt{\mathcal{J}(s_0)}-\sqrt{\mathcal{J}(s)}\lesssim b_0. \tag{2.74}$$

□

Corollary 2.21.

$$e^{-e^{\frac{5\pi}{b}}}\le\lambda,\quad \textit{which shows the other half of hypothesis (2.40) cannot fail.}$$

Proof. From the approximate dynamics of λ, equation (2.70),

$$-\frac{\lambda_s}{\lambda}\le 3b\le\frac{4\pi}{\log s}\quad\Longrightarrow\quad-\log\lambda(s)\le-\log\lambda_0+4\pi\int_{s_0}^{s}\frac{1}{\log\sigma}\,d\sigma$$

Bound the integral with $4\pi(s-s_0)$, and by using (2.72) again, $e^{4\pi(s-s_0)}\le e^{4\pi\left(e^{\frac{4\pi}{3b(s)}}-s_0\right)}$. With the definition of s_0 (2.46) and initial condition (2.36),

$$\lambda(s)\ge\lambda_0e^{4\pi s_0}\,e^{-4\pi e^{\frac{4\pi}{3b(s)}}}>e^{-e^{\frac{5\pi}{b(s)}}}.$$

□

2.4.4. *Hypothesis (2.41).* As another consequence of the approximate dynamics of λ, equation (2.70),

$$\frac{d}{ds}\left(\lambda^2e^{\frac{5\pi}{b}}\right)=2\lambda^2e^{\frac{5\pi}{b}}\left(\frac{\lambda_s}{\lambda}+b-b-\frac{5\pi b_s}{2b^2}\right)\le-\lambda^2be^{5\pi}b<0,$$
$$\Longrightarrow\lambda^2(t)e^{\frac{5\pi}{b(t)}}\le\lambda_0^2e^{\frac{5\pi}{b_0}}.$$

Then, due to the bounds on Γ_b, (2.25), and the initial condition (2.37),

$$\lambda^2(t)|E_0|<\Gamma_{b(t)}^4\,e^{\frac{5\pi}{b_0}}\,\lambda_0^2|E_0|<\Gamma_{b(t)}^4\,e^{\frac{5\pi}{b_0}}\,\Gamma_{b_0}^{10}\ll\Gamma_{b(t)}^4.$$

This shows that hypothesis (2.41) cannot fail.

2.4.5. *Dynamics of Theorem 1.1.*

Proof of Log-log Rate. By proving $T_{\mathrm{hyp}}=T_{\max}$, we have already shown blowup in finite time, due to Corollary 2.18. Here we establish the rate. By direct calculation and a change of variable,

$$-\partial_t\left(\lambda^2\log|\log\lambda|\right)=-\frac{\lambda_s}{\lambda}\log|\log\lambda|\left(2+\frac{1}{|\log\lambda|\log|\log\lambda|}\right). \tag{2.75}$$

Recall the approximate dynamic $-\frac{\lambda_s}{\lambda}\approx b$, and with hypothesis (2.40), equation (2.75) reads,

$$\frac{1}{C}\le-\partial_t\left(\lambda^2\log|\log\lambda|\right)\le C.$$

Integrate over $[t,T_{\max})$. Since λ is very small we may estimate,

$$\frac{1}{C}\left(\frac{T_{\max}-t}{\log|\log(T_{\max}-t)|}\right)^{\frac12}\le\lambda(t)\le C\left(\frac{T_{\max}-t}{\log|\log(T_{\max}-t)|}\right)^{\frac12}. \tag{2.76}$$

Moreover, the relationship between $\lambda(t)$ and the log-log rate has a universal asymptotic value as $t\to T_{\max}$, see [22, Prop 6]. □

Proof of Singularity Description in L^2. The proof here is heavily inspired by that given by Merle and Raphaël, [20, Section 4]. First, we show for any $R > 0$ there exists u^* such that,

$$\tilde{u}(t) \to u^* \quad \text{in} \quad L^2_x(|x| \geq R) \quad \text{as} \quad t \to T_{\max}. \tag{2.77}$$

Second, to establish equation (1.12), we will prove that both,

$$u^* \in L^2(\mathbb{R}^2), \quad \text{and,} \quad \int |u^*|^2 = \lim_{t \to T_{\max}} \int |\tilde{u}(t)|^2. \tag{2.78}$$

Let $\epsilon_0 > 0$ be arbitrary. Choose some $T_{\max} - \epsilon_0 < t(R) < T_{\max}$. By hypothesis (2.40) we may assume that, $u(t) = \tilde{u}$ on $\left\{|x| > \frac{R}{4}\right\}$ for $t \in [t(R), T_{\max})$ and by equation (2.74) we may assume that, $\int_{t(R)}^{T_{\max}} \int |\nabla \tilde{u}|^2 \, dx \, dt < \epsilon_0$. For a parameter $\tau > 0$, to be fixed later, we denote,

$$v^\tau(t, x) = u(t + \tau, x) - u(t, x). \tag{2.79}$$

Since $t(R) < T_{\max}$, $u(t)$ is strongly continuous in L^2 at time $t(R)$. Thus, there exists τ_0 such that,

$$\int |v^\tau(t(R))|^2 \, dx < \epsilon_0 \quad \text{for all } \tau \in [0, \tau_0]. \tag{2.80}$$

Denote ϕ_R a smooth cutoff of the region $\mathbb{1}_{\{|x|>R\}}$. By direct calculation,

$$\begin{aligned} \frac{1}{2}\partial_t \left(\int \phi_R |v^\tau|^2 \right) =& Im \left(\int \nabla \phi_R \cdot \nabla v^\tau \overline{v^\tau} \, dx \right) \\ &+ Im \left(\int \phi_R v^\tau \left(\overline{u|u|^2(t+\tau) - u|u|^2(t)} \right) dx \right). \end{aligned} \tag{2.81}$$

Regarding the first RH term of (2.81), use Hölder, (2.80), and the choice of $t(R)$,

$$\int_{t(R)}^{T_{\max}} |Im \left(\int \nabla \phi_R \cdot \nabla v^\tau \overline{v^\tau} \, dx \right) dt| \leq C \left(\int_{t(R)}^{T_{\max}} 1^2 \, dt \right)^{\frac{1}{2}} \epsilon_0^{\frac{1}{2}} < C\epsilon_0.$$

Regarding the second RH term of (2.81), control with $\|\tilde{u}\|_{L^4}^4 \leq \|\tilde{u}\|_{L^2}^2 \|\tilde{u}\|_{\dot{H}^1}^2 \ll \|\tilde{u}\|_{H^1}^2$, and integrate in time to get control by ϵ_0. We have proven that $\tilde{u}$ is Cauchy on $|x| > R$,

$$\int \phi_R |v^\tau(t)|^2 \, dx < C\epsilon_0 \quad \text{for all } \tau \in [0, \tau_0] \text{ and } t \in [t(R), T_{\max} - \tau).$$

We now turn our attention to (2.78). The profile and radiation have support of radius $R(t) = A(t)\lambda(t)$, which, due to hypothesis (2.40), is going to zero with a bound, $R(t) \leq (T_{\max} - t)^{\frac{1}{2} - \delta}$. From the definition of $A(t)$ and equation (2.45) we may bound $\int (1 - \phi_{R(t)}) |\tilde{u}(t)|^2$ to prove,

$$\lim_{t \to T_{\max}} \int \phi_{R(t)} |u(t)|^2 = \lim_{t \to T_{\max}} \int \phi_{R(t)} |\tilde{u}(t)|^2 = \lim_{t \to T_{\max}} \int |\tilde{u}(t)|^2,$$

which proves that the following limit exists,

$$\int |u^*|^2 = \lim_{t \to T_{\max}} \int \phi_{R(t)} |u(t)|^2.$$

This completes the proof of (2.78) and (1.12). □

3. The Spectral Property

We now provide a numerically assisted proof of the spectral property for the case $m = 1$. We also present some computations on higher order vortices and discuss why they do not work. Before proving the Spectral Property of Section 1.4, we will establish the following variant:

Proposition 3.1. *Let $\epsilon \in H^1_{(m)}$ satisfy the orthogonality conditions,*

$$\left\langle \epsilon_1, Q^{(m)} \right\rangle = \left\langle \epsilon_1, \Lambda Q^{(m)} \right\rangle = \left\langle \epsilon_2, \Lambda Q^{(m)} \right\rangle = \left\langle \epsilon_2, \Lambda^2 Q^{(m)} \right\rangle = 0. \tag{3.82}$$

Then, for the case $m = 1$, there is a universal constant $C_m > 0$, so that,

$$\mathcal{H}^{(m)}(\epsilon, \epsilon) \geq C_m \int \left(|\nabla_y \epsilon|^2 + e^{-|y|} |\epsilon|^2 \right) dy. \tag{3.83}$$

Proposition 1.1 is an immediate corollary[10]. Following [6, 15], we proceed in two steps. First we count the number of negative eigenvalues of the operators $\mathcal{L}_1^{(m)}$ and $\mathcal{L}_2^{(m)}$. We then show that the assumed L^2 orthogonality conditions are sufficient to project away from the negative directions of the bilinear forms, $\mathcal{H}_1^{(m)}$ and $\mathcal{H}_2^{(m)}$, associated with $\mathcal{L}_1^{(m)}$ and $\mathcal{L}_2^{(m)}$.

We now restrict ourselves to $\epsilon \in H^1_{(m)}$, $\epsilon = e^{im\theta}\epsilon_{\text{rad}}$, where ϵ_{rad} is a purely radial function,

$$\epsilon_{\text{rad}} \in H^1_{\text{rad}+} \equiv H^1_{\text{rad}}(\mathbb{R}^2) \cap \left\{u \mid |x|^{-1}u \in L^2(\mathbb{R}^2)\right\}. \tag{3.84}$$

Given $\epsilon \in H^1_{(m)}$, we calculate

$$\mathcal{L}_1^{(m)}\epsilon = e^{im\theta}\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2} + 3R^{(m)} y \cdot \nabla R^{\pm}\right)\epsilon_{\text{rad}} \tag{3.85}$$

This motivates defining the two operators and inner products on $H^1_{\text{rad}+}$

$$\mathcal{L}^{(m)}_{1,\text{rad}} \equiv -\Delta_{\text{rad}} + \frac{m^2}{r^2} + 3R^{(m)} y \cdot \nabla R^{(m)} = -\Delta_{\text{rad}} + \frac{m^2}{r^2} + \mathcal{V}_{1,\text{rad}} \tag{3.86a}$$

$$\mathcal{L}^{(m)}_{2,\text{rad}} \equiv -\Delta_{\text{rad}} + \frac{m^2}{r^2} + R^{(m)} y \cdot \nabla R^{(m)} = -\Delta_{\text{rad}} + \frac{m^2}{r^2} + \mathcal{V}_{2,\text{rad}} \tag{3.86b}$$

$$\mathcal{H}^{(m)}_{1,\text{rad}}(\cdot,\cdot) \equiv \left\langle \mathcal{L}^{(m)}_{1,\text{rad}}\cdot,\cdot\right\rangle, \quad \mathcal{H}^{(m)}_{2,\text{rad}}(\cdot,\cdot) \equiv \left\langle \mathcal{L}^{(m)}_{2,\text{rad}}\cdot,\cdot\right\rangle, \tag{3.86c}$$

where Δ_{rad} is the radial Laplacian, $\Delta_{\text{rad}} \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$.

The orthogonality conditions, (3.82), are now formulated as

$$\left\langle \epsilon_1^{\text{rad}}, R^{(m)}\right\rangle = \left\langle \epsilon_1^{\text{rad}}, \Lambda R^{(m)}\right\rangle = \left\langle \epsilon_2^{\text{rad}}, \Lambda R^{(m)}\right\rangle = \left\langle \epsilon_2^{\text{rad}}, \Lambda^2 R^{(m)}\right\rangle = 0, \tag{3.87}$$

where

$$\epsilon = e^{im\theta}\left(\epsilon_1^{\text{rad}} + i\epsilon_2^{\text{rad}}\right)$$

and $\epsilon \in H^1_{(m)}$, $\epsilon_j^{\text{rad}} \in H^1_{\text{rad}+}$.

All that follows relies on the reduction to a series of one dimensional radial problems.

3.1. **The Index of Bilinear Forms.**

Definition 3.2. *The index of a bilinear form $\mathcal{B}$ with respect to vector space V, denoted* $\text{ind}_V\mathcal{B}$*, is the minimal co-dimension over all subspaces of V on which $\mathcal{B}$ is a positive.*

For bilinear forms induced by self-adjoint operators (*i.e.* $\mathcal{B} = \langle \mathcal{L}\cdot,\cdot\rangle$), the index corresponds to the number of negative eigenvalues of the operator. To calculate the index, we extend Theorem XIII.8 of Reed & Simon [29] to:

Theorem 3.1. *Let U solve,*

$$\mathcal{L}\,U = -\frac{d^2}{dr^2}U - \frac{1}{r}\frac{d}{dr}U + \mathcal{V}(r)U + \frac{m^2}{r^2}U = 0,$$

with initial conditions given by the limits,

$$\lim_{r\to 0} r^{-m}U(r) = 1, \quad \lim_{r\to 0}\frac{d}{dr}\left(r^{-m}U(r)\right) = 0,$$

and where the potential $\mathcal{V}$ is sufficiently smooth and decaying at ∞. Then, the number of roots of U not at the origin, $N(U)$, is finite and equal to the index of the bilinear form associated to $\mathcal{L}$ over the vector space $H^1_{\text{rad}+}$.

Proof. The proof, which we omit, is quite similar to the proof of the indicated Theorem of Reed & Simon. In turn, that proof is a generalization of the Sturm Oscillation theorem for two point boundary value problems. □

[10] See the end of Subsection 3.3 for details.

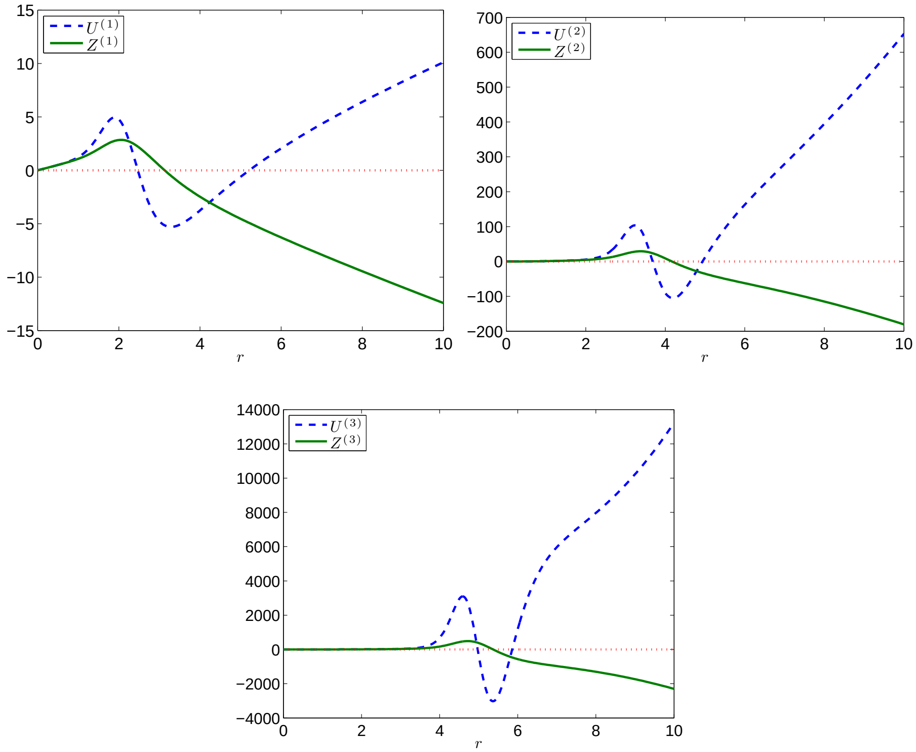


FIGURE 3. Plots of the functions $U^{(m)}$ and $Z^{(m)}$ solving (3.89) for $m = 1, 2, 3$. The number of non-zero zero crossings determines the indexes of H_1 and H_2.

Proposition 3.3 (Numerically Verified)**.** *For the cases* $m = 1, 2, 3$*,*

$$\operatorname{ind}_{H^1_{\mathrm{rad}+}} \mathcal{H}^{(m)}_{1,\mathrm{rad}} = 2, \quad \textit{and}, \quad \operatorname{ind}_{H^1_{\mathrm{rad}+}} \mathcal{H}^{(m)}_{2,\mathrm{rad}} = 1. \tag{3.88}$$

Proof. Using the methods described in Appendix B, we solve,

$$\mathcal{L}^{(m)}_{1,\mathrm{rad}} U^{(m)} = 0, \quad \lim_{r\to 0} r^{-m} U^{(m)}(0) = 0, \quad \lim_{r\to 0} \frac{d}{dr} r^{-m} U^{(m)}(r) = 0, \tag{3.89a}$$

$$\mathcal{L}^{(m)}_{2,\mathrm{rad}} Z^{(m)} = 0, \quad \lim_{r\to 0} r^{-m} Z^{(m)}(0) = 0, \quad \lim_{r\to 0} \frac{d}{dr} r^{-m} Z^{(m)}(r) = 0. \tag{3.89b}$$

Plotting the solutions in Figure 3, we can see that $U^{(m)}$ has two zero crossings and $Z^{(m)}$ has one zero crossing. Subject to the acceptance of these computations, Theorem 3.1 yields the result. □

Proposition 3.4. *There exists a constant,* $\delta_0 > 0$*, depending on* m*, such that for* $\delta \in (0, \delta_0)$*, the bilinear forms associated with the perturbed operators,*

$$\overline{\mathcal{L}}^{(m)}_{j,\mathrm{rad}} \equiv \mathcal{L}^{(m)}_{j,\mathrm{rad}} - \delta e^{-|y|},$$

have the same index, i.e.

$$\operatorname{ind} \mathcal{H}^{(m)}_{j,\mathrm{rad}} = \operatorname{ind} \overline{\mathcal{H}}^{(m)}_{j,\mathrm{rad}}.$$

Proof. We briefly sketch the proof, which follows from three observations. First, the of the solutions of the perturbed form of (3.89) are continuous with respect to δ. In particular, there is C^1_{loc} convergence. Second, the roots of the index functions, in the perturbed and unperturbed cases, must be simple. For a sufficiently small δ_0, we can ensure that on any compact interval the perturbed and unperturbed solutions have the same number of zeros. Finally, for a sufficiently large compact interval, outside the interval the equation

Table 1. Inner products associated with $\mathcal{L}_{1,\text{rad}}^{(m)}$ for different winding numbers.

m	$K_1^{(m)}$	$K_2^{(m)}$	$K_3^{(m)}$	$K_1^{(m)}K_2^{(m)} - \left(K_3^{(m)}\right)^2$
1	-0.48237	-25.798	1.28129	10.8025
2	0.520152	-13.1545	1.7983	-10.0762
3	2.59249	5.1232	-1.54694	10.8888

Table 2. Inner products associated with $\mathcal{L}_{2,\text{rad}}^{(m)}$ for different winding numbers.

m	$J_1^{(m)}$	$J_2^{(m)}$	$J_3^{(m)}$	$J_1^{(m)}J_2^{(m)} - \left(J_3^{(m)}\right)^2$
1	6.6985	163.548	-47.7764	-1.1871e+03
2	25.1685	1319.28	-235.186	-2.2108e+04
3	82.6396	8426.22	-936.752	-1.8116e+05

is approximately "free" (the localized potentials are negligible), and we can ensure there are no additional zeros; this may require further shrinking δ_0. □

3.2. **Orthogonality Conditions and Inner Products.** To verify that orthogonality conditions (3.87) project away from the negative subspaces, we need to compute a number of inner products of the form $\left\langle \mathcal{L}_{j,\text{rad}}^{(m)} u, u\right\rangle$, where u solves $\mathcal{L}_{j,\text{rad}}^{(m)} u = f$. Although these products are computed numerically, we justify their existence:

Proposition 3.5 (Numerically Verified). *Let f be a continuous, radially symmetric, localized function satisfying the bound $|f(r)| \leq Ce^{-\kappa r}$ for some positive constants C and κ. There exists a unique radially symmetric solution,*

$$\overline{\mathcal{L}}_{j,\text{rad}}^{(m)} u = f, \quad j = 1, 2.$$

that belongs to the class, $u \in L^\infty([0,\infty)) \cap C^2([0,\infty))$.

Proof. This is Proposition 2 and 4 of [6], along with our computations of the indexes in Lemma 3.89. See [15] for some additional details and a full proof in dimension $d = 1$. □

Remark 3.2. The solutions in Proposition 3.5 may not vanish as $r \to \infty$. Indeed, they can only be expected to be bounded.

Proposition 3.6 (Numerically Verified). *Let U_1, U_2, Z_1, and Z_2 be L^∞ radially symmetric functions solving,*

$$\mathcal{L}_{1,\text{rad}}^{(m)} U_1 = R^{(m)}, \tag{3.90a}$$

$$\mathcal{L}_{1,\text{rad}}^{(m)} U_2 = \Lambda R^{(m)}, \tag{3.90b}$$

$$\mathcal{L}_{2,\text{rad}}^{(m)} Z_1 = \Lambda R^{(m)}, \tag{3.90c}$$

$$\mathcal{L}_{2,\text{rad}}^{(m)} Z_2 = \Lambda^2 R^{(m)}. \tag{3.90d}$$

Then the inner products,

$$K_1^{(m)} \equiv \left\langle \mathcal{L}_{1,\text{rad}}^{(m)} U_1, U_1 \right\rangle, \qquad K_2^{(m)} \equiv \left\langle \mathcal{L}_{1,\text{rad}}^{(m)} U_2, U_2 \right\rangle, \quad K_3^{(m)} \equiv \left\langle \mathcal{L}_{1,\text{rad}}^{(m)} U_1, U_2 \right\rangle,$$
$$J_1^{(m)} \equiv \left\langle \mathcal{L}_{2,\text{rad}}^{(m)} Z_1, Z_1 \right\rangle, \qquad J_2^{(m)} \equiv \left\langle \mathcal{L}_{2,\text{rad}}^{(m)} Z_2, Z_2 \right\rangle, \quad J_3^{(m)} \equiv \left\langle \mathcal{L}_{2,\text{rad}}^{(m)} Z_1, Z_2 \right\rangle,$$

take the values given in Tables 1 and 2.

Proof. Using the methods described in Appendix B, these are computed numerically. □

As with the indices, we have stability of the inner products with respect to perturbation by a small portential:

Proposition 3.7. *Let $\overline{U}_l$ and $\overline{Z}_l$ denote the solutions and $\overline{K}_l^{(m)}$ and $\overline{J}_l^{(m)}$ the inner products, analogous to those of Proposition 3.6, for the boundary value problems with the perturbed operators, $\overline{\mathcal{L}}_{j,\mathrm{rad}}^{(m)}$. For $\delta_0 > 0$ sufficiently small, the solutions and inner products are continuous with respect to δ.*

Proof. This follows from the invertiblity and continuity with respect to δ of the operators. $\square$

3.3. **Proof of the Spectral Property.** We are now able to prove Proposition 3.1. The arguement closely follows the proofs found in [6, 15]. The two bilinear forms, $\overline{\mathcal{H}}_1^{(m)}$ and $\overline{\mathcal{H}}_2^{(m)}$, are treated seperately. First, we will show that L^2 orthogonality to $Q^{(m)}$ and $\Lambda Q^{(m)}$ suffices to project away from the negative subspace of $\mathcal{H}_1^{(m)}$. This will only be successful for $m = 1$. Later, we will show that L^2 orthogonality to $\Lambda Q^{(m)}$ and $\Lambda^2 Q^{(m)}$ projects away from the negative subspace of $\mathcal{H}_2^{(m)}$.

Spectral Property for $\mathcal{H}_1^{(m)}$. Given an element $u \in H^1_{(m)}$, $u = e^{im\theta}u_{\mathrm{rad}}$, satisfying orthogonality conditions (3.82), showing positivity of $\mathcal{H}_1^{(m)}$ on such a u is equivalent to showing posiviity of $\mathcal{H}_{1,\mathrm{rad}}^{(m)}$ on $u_{\mathrm{rad}} \in H^1_{\mathrm{rad}+}$ satisfying orthogonality conditions (3.87).

By Propositions 3.3 and 3.4, $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(m)}$ has a two-dimensional subspace of negative directions. Recall the notation of equation (3.90). Let $\mathrm{V} = \mathrm{span}\left\{\overline{U}_1, \overline{U}_2\right\}$. We will prove that, for $m = 1$, $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(m)}$ is negative on all of V. Indeed, consider an arbitrary element of this space,

$$\hat{U} = c_1\overline{U}_1 + c_2\overline{U}_2,$$

and compute,

$$\begin{aligned} \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(m)}(\hat{U}, \hat{U}) &= c_1^2\overline{K}_1^{(m)} + 2c_1c_2\overline{K}_3^{(m)} + c_2^2\overline{K}_2^{(m)} \\ &= \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} \overline{K}_1^{(m)} & \overline{K}_3^{(m)} \\ \overline{K}_3^{(m)} & \overline{K}_2^{(m)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned} \tag{3.91}$$

If the above matrix is negative definite, then the bilinear form is negative on the two dimensional space V. We examine the matrix using the computations in Table 1 and elementary properties of matrices. For $m = 1$,

$$\mathrm{tr} = \overline{K}_1^{(1)} + \overline{K}_2^{(1)} = -26.2804 + \mathrm{o}(1),$$

where o(1) corresponds to taking the perturbation parameter, δ, sufficiently small. Therefore the sum of the two eigenvalues is negative; at least one is negative. Next,

$$\det = \overline{K}_1^{(1)}\overline{K}_2^{(1)} - (\overline{K}_3^{(1)})^2 = 10.8025 + \mathrm{o}(1),$$

so the two eigenvalues have the same sign. Therefore $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}$ is negative on V. Table 1 shows that this is false for $m = 2, 3$. We restrict our attention to $m = 1$.

Pretending that $\mathrm{V} \subset H^1_{\mathrm{rad}+}(\mathbb{R}^2)$, we could decompose the space as

$$H^1_{\mathrm{rad}+}(\mathbb{R}^2) = \mathrm{V} \oplus_{\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}} \mathrm{V}^\perp \tag{3.92}$$

where our notation indicates that we have formed the orthogonal complement with respect to the $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}$ bilinear form. The non-degeneracy of the matrix (3.91) justifies this decomposition.

It follows that $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}$ is positive on $\mathrm{V}^\perp$. Otherwise, there would be $W \in \mathrm{V}^\perp$ such that $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(W, W) < 0$, which implies by construction that, $\mathrm{span}\left\{W, \overline{U}_1, \overline{U}_2\right\}$, is a negative definite space of $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}$ with dimension three. But then, given any subspace $\mathrm{U} \subset H^1_{\mathrm{rad}+}$ of codimension two, $\mathrm{U} \cap \mathrm{span}\left\{W, \overline{U}_1, \overline{U}_2\right\} \neq \emptyset$, which contradicts the index calculation.

Finally, given any function $u \in H^1_{\mathrm{rad}+}$ and L^2 orthogonal to $R^{(1)}$ and $\Lambda R^{(1)}$, we decompose u as

$$u = c_1\overline{U}_1 + c_2\overline{U}_2 + u^\perp$$

where, $u^{\perp} \in \mathrm{V}^{\perp}$, again in the sense of (3.92). Then,

$$
\begin{aligned}
0 = \left\langle u, R^{(1)} \right\rangle_{L^2} &= c_1 \left\langle \overline{U}_1, R^{(1)} \right\rangle_{L^2} + c_2 \left\langle \overline{U}_2, R^{(1)} \right\rangle_{L^2} + \left\langle u^{\perp}, R^{(1)} \right\rangle_{L^2} \\
&= c_1 \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(\overline{U}_1, \overline{U}_1) + c_2 \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(\overline{U}_2, \overline{U}_1) + \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(u^{\perp}, \overline{U}_1) \\
&= c_1 \overline{K}_1^{(1)} + c_2 \overline{K}_3^{(1)}, \\
0 = \left\langle u, \Lambda R^{(1)} \right\rangle_{L^2} &= c_1 \left\langle \overline{U}_1, \Lambda R^{(m)} \right\rangle_{L^2} + c_2 \left\langle \overline{U}_2, \Lambda R^{(1)} \right\rangle_{L^2} + \left\langle u^{\perp}, \Lambda R^{(1)} \right\rangle_{L^2} \\
&= c_1 \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(\overline{U}_1, \overline{U}_2) + c_2 \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(\overline{U}_2, \overline{U}_2) + \overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(u^{\perp}, \overline{U}_2) \\
&= c_1 \overline{K}_3^{(1)} + c_2 \overline{K}_2^{(1)}.
\end{aligned}
$$

Due to the non-degeneracy of (3.91), the only solution is $c_1 = c_2 = 0$. Therefore, for all such u,

$$\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(1)}(u, u) \geq 0.$$

This yields the positivity of $\overline{\mathcal{H}}_1^{(1)}$ on $H^1_{(1)}$.

Of course, $\overline{U}_1$ and $\overline{U}_2$ are *not* in $H^1_{\mathrm{rad}+}$. The above argument is made rigorous by introducing an appropriate cutoff function and then taking limits. We refer the reader to [6, 15]; we will not reproduce this here.

□

Spectral Property for $\mathcal{H}_2^{(m)}$. As in the case of $\overline{\mathcal{H}}_1^{(m)}$, we will prove positivity of $\overline{\mathcal{H}}_2^{(m)}$ subject to the orthogonality conditions, by working with the associated radial form, $\overline{\mathcal{H}}_{2,\mathrm{rad}}^{(m)}$. By Propositions 3.3 and 3.4, $\overline{\mathcal{H}}_{2,\mathrm{rad}}^{(m)}$ has one negative direction. Examing Table 2, neither $\overline{Z}_1^{(m)}$ nor $\overline{Z}_2^{(m)}$ appears to point in the negative direction. Define,

$$\hat{R}^{(m)} \equiv \Lambda R^{(m)} - \frac{\overline{J}_3^{(m)}}{\overline{J}_2^{(m)}} \Lambda^2 R^{(m)}, \tag{3.93a}$$

$$\hat{Z} \equiv \overline{Z}_1 - \frac{\overline{J}_3^{(m)}}{\overline{J}_2^{(m)}} \overline{Z}_2. \tag{3.93b}$$

Then $\mathcal{L}_2^{(m)} \hat{Z} = \hat{R}^{(m)}$ and,

$$\overline{\mathcal{H}}_2^{(m)}(\hat{Z}, \hat{Z}) = \frac{1}{\overline{J}_2^{(m)}} \left(\overline{J}_1^{(m)} \overline{J}_2^{(m)} - \left(\overline{J}_3^{(m)} \right)^2 \right) < 0. \tag{3.94}$$

Now that we have constructed a negative direction, we apply a similar argument as in the case of $\overline{\mathcal{H}}_{1,\mathrm{rad}}^{(m)}$; however, this will hold not just for $m = 1$, but also for $m = 2, 3$. We decompose $H^1_{\mathrm{rad}+}(\mathbb{R}^2)$ as

$$H^1_{\mathrm{rad}+}(\mathbb{R}^2) = \mathrm{span}\left\{ \hat{Z} \right\} \oplus_{\overline{\mathcal{H}}_{2,\mathrm{rad}}^{(m)}} \mathrm{span}\left\{ \hat{Z} \right\}^{\perp} \tag{3.95}$$

Since the index of $\overline{\mathcal{H}}_{2,\mathrm{rad}}^{(m)}$ is one, we are assured that it is positive on $\mathrm{span}\left\{ \hat{Z} \right\}^{\perp}$. Finally, given $v \in H^1_{\mathrm{rad}+}$ orthogonal to $\Lambda R^{(m)}$ and $\Lambda^2 R^{(m)}$, it may be decomposed as $v = c_1 \hat{Z} + v^{\perp}$, and,

$$
\begin{aligned}
0 = \left\langle v, \hat{R}^{(m)} \right\rangle_{L^2} &= c_1 \overline{\mathcal{H}}_2^{(m)}(\hat{Z}, \hat{Z}) + \overline{\mathcal{H}}_2^{(m)}(v^{\perp}, \hat{Z}) \\
&= c_1 \overline{\mathcal{H}}_2^{(m)}(\hat{Z}, \hat{Z}).
\end{aligned}
$$

Invoking (3.94), this implies that, $v = v^{\perp} \in \mathrm{span}\left\{ \hat{Z} \right\}^{\perp}$. Therefore, for such v,

$$\overline{\mathcal{H}}_{2,\mathrm{rad}}^{(m)}(v, v) \geq 0$$

for $m = 1, 2, 3$. Posivitiy of $\overline{\mathcal{H}}_2^{(m)}$ on $H^1_{(m)}$, subject to orthgonality to $\Lambda Q^{(m)}$ and $\Lambda^2 Q^{(m)}$, follows. □

Proof of Proposition 3.1. Given $\epsilon = \epsilon_1 + i\epsilon_2$ satisfying the orthogonality conditions of Proposition 3.1 we have proven that,

$$\overline{\mathcal{H}}^{(1)}(\epsilon,\epsilon) = \overline{\mathcal{H}}_1^{(1)}(\epsilon_1,\epsilon_1) + \overline{\mathcal{H}}_2^{(1)}(\epsilon_2,\epsilon_2) \geq 0,$$

from which we infer,

$$\mathcal{H}^{(1)}(\epsilon,\epsilon) \geq \delta \int e^{-|y|}|\epsilon|^2 dy.$$

Let $\theta > 0$. Then,

$$(1+\theta)\mathcal{H}^{(1)}(\epsilon,\epsilon) \geq \theta \int |\nabla\epsilon|^2 dy + \theta \int \mathcal{V}_1|\epsilon_1|^2 + \mathcal{V}_2|\epsilon_2|^2 dy + \delta \int e^{-|y|}|\epsilon|^2 dy.$$

Although the potentials are sign indefinite, for θ sufficiently small,

$$\theta \int \mathcal{V}_1|\epsilon_1|^2 + \mathcal{V}_2|\epsilon_2|^2 dy + \delta \int e^{-|y|}|\epsilon|^2 dy \geq \frac{\delta}{2} \int e^{-|y|}|\epsilon|^2 dy. \tag{3.96}$$

We now have the result,

$$\begin{aligned}
\mathcal{H}^{(1)}(\epsilon,\epsilon) &\geq \frac{\theta}{1+\theta} \int |\nabla\epsilon|^2 dy + \frac{\delta}{2(1+\theta)} \int e^{-|y|}|\epsilon|^2 dy \\
&\geq \delta_0 \int |\nabla\epsilon|^2 + e^{-|y|}|\epsilon|^2 dy.
\end{aligned}$$

□

Proof of Proposition 1.1. Let $\epsilon \in H^1_{(1)}(\mathbb{R}^2)$ with $\epsilon = \epsilon_1 + i\epsilon_2$, and further decompose this as:

$$\epsilon_1 = e^{i\theta}\left(u + c_1 R^{(1)} + c_2 \Lambda R^{(1)}\right), \tag{3.97a}$$

$$\epsilon_2 = e^{i\theta}\left(v + d_1 \Lambda R^{(1)} + d_2 \Lambda^2 R^{(1)}\right), \tag{3.97b}$$

where $u \perp_{L^2} R^{(1)}, \Lambda R^{(1)}$ and $v \perp_{L^2} \Lambda R^{(1)}, \Lambda^2 R^{(1)}$. Expaning,

$$\begin{aligned}
\mathcal{H}^{(m)}(\epsilon,\epsilon) &= \mathcal{H}_1^{(m)}(\epsilon_1,\epsilon_1) + \mathcal{H}_2^{(m)}(\epsilon_2,\epsilon_2), \\
\mathcal{H}_1^{(m)}(\epsilon_1,\epsilon_1) &= \mathcal{H}_{1,\mathrm{rad}}^{(m)}(u,u) + 2c_1\left\langle \mathcal{L}_{1,\mathrm{rad}}^{(m)} u, R^{(1)}\right\rangle + 2c_2\left\langle \mathcal{L}_{1,\mathrm{rad}}^{(m)} u, \Lambda R^{(m)}\right\rangle \\
&\quad + c_1^2 M_1^{(m)} + c_2^2 M_2^{(m)} + 2c_1c_2 M_3^{(m)}, \\
\mathcal{H}_2^{(m)}(\epsilon_2,\epsilon_2) &= \mathcal{H}_{2,\mathrm{rad}}^{(m)}(v,v) + 2d_1\left\langle \mathcal{L}_{2,\mathrm{rad}}^{(m)} v, \Lambda R^{(m)}\right\rangle + 2d_2\left\langle \mathcal{L}_2^{(m)} v, \Lambda^2 R^{(m)}\right\rangle \\
&\quad + d_1^2 N_1^{(m)} + d_2^2 N_2^{(m)} + 2d_1d_2 N_3^{(m)},
\end{aligned}$$

where $M_j^{(m)}, N_j^{(m)}$ are fixed terms arising from applications of the $\mathcal{H}_{j,\mathrm{rad}}^{(m)}$ bilinear forms to combinations of $R^{(m)}$, $\Lambda R^{(m)}$, and $\Lambda^2 R^{(m)}$.

We now construct a lower bound. Let $\theta > 0$. Then

$$\begin{aligned}
c_1\left\langle \mathcal{L}_1^{(m)} u, R^{(m)}\right\rangle &\leq \frac{1}{2}\left(\theta^{-2}c_1^2 + \theta^2\left\langle \mathcal{L}_{1,\mathrm{rad}}^{(m)} u, R^{(m)}\right\rangle^2\right) \\
&\leq \frac{1}{2}\left[\theta^{-2}c_1^2 + \theta^2\left(\int |u||\mathcal{L}_{1,\mathrm{rad}}^{(m)} R^{(m)}|\right)^2\right] \\
&\leq \frac{1}{2}\left[\theta^{-2}c_1^2 + \theta^2\left(\int |u||\mathcal{L}_{1,\mathrm{rad}}^{(m)} R^{(m)}|^{1/2}|\mathcal{L}^{(1)} R^{(m)}|^{1/2}\right)^2\right] \\
&\leq C\left[\theta^{-2}c_1^2 + \theta^2 \int |\mathcal{L}_{1,\mathrm{rad}}^{(m)} R^{(m)}||u|^2\right] \\
&\leq C\left(\theta^{-2}c_1^2 + \theta^2 \int e^{-|y|}|u|^2\right).
\end{aligned} \tag{3.98}$$

The other terms in which u or v appears once are similarly controlled. Therefore,

$$\mathcal{H}^{(m)}(\epsilon,\epsilon) \geq \mathcal{H}^{(m)}_{1,\mathrm{rad}}(u,u) + \mathcal{H}^{(m)}_{2,\mathrm{rad}}(v,v) - C\theta^{-2}(c_1^2+c_2^2+d_1^2+d_2^2) - C\theta^2 \int e^{-|y|}|u+iv|^2,$$

For the case $m=1$, we apply Proposition 3.1 to get

$$\begin{aligned}\mathcal{H}^{(1)}(\epsilon,\epsilon) \geq & C_{(1)} \int |\nabla(u+iv)|^2 \\ & + \left(C_{(1)} - C\theta^2\right) \int e^{-|y|}|u+iv|^2 - C\theta^{-2}(c_1^2+c_2^2+d_1^2+d_2^2) \\ \geq & \frac{C_{(1)}}{2} \int |\nabla(u+iv)|^2 + e^{-|y|}|u+iv|^2 - C\theta^{-2}(c_1^2+c_2^2+d_1^2+d_2^2),\end{aligned}$$

where we take $\theta > 0$ sufficienty small. Finally,

$$\begin{aligned}\int |\nabla e^{i\theta}(u+iv)|^2 + e^{-|y|}|e^{i\theta}(u+iv)|^2 & \\ & \geq C\left(\int |\nabla\epsilon|^2 + e^{-|y|}|\epsilon|^2\right) - \mathrm{O}\left(c_1^2+c_2^2+d_1^2+d_2^2\right).\end{aligned}$$

□

Appendix A. Almost-Self Similar Profiles

In this Appendix, we outline the proof of Proposition 2.1, showing modifications of the proof given in the case $m=0$, [18, 19, 22]. We then briefly discuss the proof of Proposition 2.3. Recall that for $e^{ib\frac{r^2}{4}}Q_b^{(m)} = e^{im\theta}P_b^{(m)}(r)$ we have equation (2.18),

$$\Delta P_b^{(m)} - \left(1 + \frac{m^2}{r^2} - \frac{b^2}{4}r^2\right)P_b^{(m)} + P_b^{(m)}|P_b^{(m)}|^2 = 0.$$

This is not a scale-invariant equation, and there is no clear representative solution. Fibich and Gavish [4] chose to consider the solution where the boundary condition $\lim_{r\to 0} r^{-m}P_b^{(m)}(r) \neq 0$ is chosen to minimize the amplitude of the asymptotic oscillation. Since we intend to truncate anyways, it is more convenient to choose boundary conditions,

$$P_b^{(m)}(r)\begin{cases} \neq 0 & \text{for} \quad 0 < r < (1-\eta)R_b, \\ = 0 & \text{for} \quad\quad\quad r = (1-\eta)R_b. \end{cases} \tag{A.99}$$

Recall that R_b was chosen, (2.19), so that the strong maximum principle applies to, $\Delta - \left(1 + \frac{m^2}{r^2} - \frac{b^2}{4}r^2\right)$, on a region larger than, $r \leq (1-\eta)R_b$.

Step 1: Existence of $P_b^{(m)}$.

Following the argument of [18, p605-606], let $\mathcal{F}_{(m)}$ denote the space of radial profiles of functions in $H^1_{(m)}$. That is, radial H^1 functions $f(x)$ for which $x^{-1}f(x) \in L^2$. Perform a constrained minimization of,

$$2\,J_b[w] = \int |\nabla w|^2 + \int |w|^2 + m^2 \int |\frac{w}{r}|^2 - \frac{b^2}{4}\int |rw|^2,$$

over the subspace of finite-variance functions in $\mathcal{F}_{(m)}$ with $w((1-\eta)R_b) = 0$ and $\int |w|^4 = 1$, where all integrals are taken over a larger compact set, for example $r \leq (1-\eta^2)R_b$. Note that J_b is coercive on $H^1_{(m)}(\mathbb{R}^2)$,

$$J_b[w] \geq C(\eta)\left(\int |\nabla w|^2 + \int |w|^2 + m^2 \int |\frac{w}{r}|^2\right). \tag{A.100}$$

This minimizing sequence can be assumed to converge weakly in $H^1_{(m)}$, which is simply a subspace of $H^1(\mathbb{R}^2)$, and thus strongly in L^4 due to Sobolev embedding on a compact domain. Here we use that equation (1.1) is energy subcritical. The Lagrange multiplier of the Frechet derivative shows that (2.18) is satisfied. Interior

regularity estimates show that the weak limit is C^3 on $r < (1-\eta)R_b$. The weak limit is also strictly positive due to $w((1-\eta)R_b) = 0$ and the maximum principle.
Step 2: L^∞ Estimates, Uniform in b.
There exists a fixed constant $C > 0$ for all $|b| > 0$ sufficiently small so that,

$$|P_b^{(m)}|_{L^\infty} \le C. \tag{A.101}$$

Moreover, there is uniform decay of the tail of the solutions. For the same b,

$$\sup_{|b|\sim 0} |P_b^{(m)}|_{L^\infty(r>R)} \longrightarrow 0 \text{ as } R \to +\infty. \tag{A.102}$$

Both bounds are proven in [18, p606]. Equation (A.101) is a simple consequence of the variational characterization of Step 1, whereas to prove equation (A.102), truncate to $r > R$, treat $r^{\frac{N-1}{2}}|P_b^{(m)}|$ as a one-dimensional function, and control by the standard Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.
Step 3: Local Convergence to $R^{(m)}$ (in C^3).
As $b \to 0$, $P_b^{(m)}$ converges weakly to some positive radial function P, with decay to 0 as $r \to +\infty$, and which satisfies, $\Delta P - \left(1 + \frac{m^2}{r^2}\right) P + P|P^2|$. This characterizes P as the unique groundstate $R^{(m)}$, [23]. Moreover, due to interior regularity estimates, on any compact set the convergence of $P_b^{(m)}$ is strong in C^3, up to a subsequence in b.
Step 4: Uniform Convergence to $R^{(m)}$ (in C^3 with exponential weight)
Here we adapt the argument of [19, p658-659]. Consider the operator $\mathcal{K} = \Delta - \left(1 + \frac{m^2}{r^2} - \frac{b^2}{4}r^2 - \frac{\eta^2}{2}\right)$, which satisfies the maximum principle on $1 < r < (1-\eta)R_b$, for any $\eta > 0$ sufficiently small. Restate (2.18) as,

$$\mathcal{K}P_b^{(m)} = \frac{\eta^2}{2}P_b^{(m)} - \left(P_b^{(m)}\right)^3. \tag{A.103}$$

Consider the new function $f_b(r) = e^{-(1-\eta)R_b\Theta\left(\frac{r}{R_b}\right)}$, with,

$$\Theta(\xi) = \mathbb{1}_{0<\xi<1} \int \sqrt{1-z^2}\, dz + \mathbb{1}_{1\le\xi}\, \Theta(1)\, \xi.$$

Note the dependence on m. By direct calculation,

$$\begin{aligned} f_b^{-1}\mathcal{K}f_b = (1-\eta)\frac{\frac{r}{R_b^2}}{\sqrt{1-\left(\frac{r}{R_b}\right)^2}} + (1-\eta)^2\left(1 - \left(\frac{r}{R_b}\right)^2\right) \\ - \frac{1}{r}\sqrt{1-\left(\frac{r}{R_b}\right)^2} - \left(1 + \frac{m^2}{r^2} - \frac{b^2}{4}r^2 - \frac{\eta^2}{2}\right). \end{aligned}$$

We now approximate each term on the region $\frac{1}{\eta} < r \le (1-\eta)R_b$,

$$\begin{aligned} f_b^{-1}\mathcal{K}f_b \le \frac{(1-\eta)^2}{R_b} + \left((1-\eta)^2 - 1\right)\left(1 - \left(\frac{r}{R_b}\right)^2\right) + \left(\frac{b^2}{4} - \frac{1}{R_b^2}\right)r^2 \\ - \eta^{\frac{3}{2}}\sqrt{2-\eta} - m^2\eta^2 + \frac{\eta^2}{2}. \end{aligned}$$

Recall that, $R_b = \frac{\sqrt{2+2\sqrt{1+b^2m^2}}}{b}$. By assuming $b > 0$ is sufficiently small with respect to η, we conclude $f_b^{-1}\mathcal{K}f_b$ is strictly negative for the given range of r.
From Step 2, and the exponential decay of $R^{(m)}$, there exists a fixed value $r_0 > \frac{1}{\eta}$ such that for all $b > 0$ sufficiently small,

$$\frac{\eta^2}{2}P_b^{(m)} - \left(P_b^{(m)}\right)^3 > 0 \quad \text{for} \quad r \in \Omega = r_0 < r < (1-\eta)R_b.$$

We have shown that $\mathcal{K}\left(c\, f_b - P_b^{(m)}\right) < 0$ for $r \in \Omega$ and any arbitrary constant $c > 0$. Now we note that,

$$\lim_{b\to 0} f_b(r_0) = e^{-(1-\eta)r_0} > 0,$$

so that we may choose our constant $c = 2R^{(m)}(r_0)e^{+(1-\eta)r_0}$ and, with our boundary condition (A.99), conclude that,

$$c\, f_b(r) - P_b^{(m)}(r)\Big|_{\partial\Omega} > 0.$$

The maximum principle may now be applied. The same argument can be applied to $R^{(m)}$, $b = 0$, and the weight $f(r) = e^{-(1-\eta)r}$. With Step 3, this proves the first precursor of (2.22),

$$\|e^{(1-C\eta)R_b\Theta\left(\frac{r}{R_b}\right)}\left(P_b^{(m)} - R^{(m)}\right)\|_{C^3} \longrightarrow 0 \quad \text{as} \quad b \to 0. \tag{A.104}$$

To prove the bound for the energy, (2.24), note that without loss of generality $(1+C\eta)(1-a) = (1-\delta) < 1$. Introduce a new operator $\mathcal{K}$ and function f_b in terms of δ in place of η and argue Step 4 again. In particular, we may assume that $r_0 < \delta R_b \ll (1-\eta)^2 R_b < r < (1-\eta)R_b \ll (1-\delta)R_b$.

Step 5: Uniqueness of $P_b^{(m)}$; Continuity in b

For fixed $b_0 > 0$ sufficiently small, and $b \approx b_0$, consider,

$$T_{b,b_0} = \left(\frac{R_b}{R_{b_0}}\right) P_b^{(m)}\left(\frac{R_b}{R_{b_0}} r\right) \tag{A.105}$$

Then $T_{b,b_0} \in \mathcal{F}_{(m)}$ and vanishes for $r = (1-\eta)R_{b_0}$, and we consider the differential, $T_\Delta = T_{b,b_0} - P_{b_0}^{(m)}$, with the same domain. The goal is to prove,

$$\|e^{im\theta}T_\Delta(r)\|_{H^1(\mathbb{R}^2)} \le C\frac{|b-b_0|}{b_0}, \tag{A.106}$$

for some fixed constant C. To do so, consider the equation for T_Δ written as,

$$\begin{aligned}\left(L_+^{(m)} - \frac{b_0^2}{4}r^2\right)T_\Delta = &-\left(\left(1-\frac{R_b^2}{R_{b_0}^2}\right)\left(1-\frac{b_0^2}{4}r^2\right) + \frac{R_b^2}{R_{b_0}^2}\frac{b_0^2 - b^2\frac{R_b^2}{R_{b_0}^2}}{4}r^2\right)T_\Delta \\ &- 3R_{(m)}^2 T_\Delta + \left(T_\Delta + P_{b_0}^{(m)}\right)^3 - \left(P_{b_0}^{(m)}\right)^3 \\ &+ \left(\left(1-\frac{R_b^2}{R_{b_0}^2}\right)\left(1-\frac{b_0^2}{4}r^2\right) + \frac{R_b^2}{R_{b_0}^2}\frac{b_0^2 - b^2\frac{R_b^2}{R_{b_0}^2}}{4}r^2\right)P_{b_0}^{(m)},\end{aligned} \tag{A.107}$$

where $L_+^{(m)}$ is the linerized operator from equation (1.10). We will use $\mathcal{F}_{b,b_0}$ to denote the final right hand term of (A.107). Note that in the case $m = 0$, and thus $R_b = \frac{2}{b}$, the final multiples of T_Δ and $P_{b_0}^{(m)}$ collapse. All three right hand terms of (A.107) are bounded in the same way as in [18, p609], with only minor adaptations[11]. To conclude the argument from [18] and establish (A.106) there only remains to show the following Lemma:

Lemma A.1. *Let $\mu_+ < 0$ be the lowest eigenvalue of $L_+^{(m)}$, and $\phi_+ \in L^2$ the corresponding normalized eigenvector. For $b > 0$ sufficiently small with respect to η, and assuming $\eta > 0$ is itself sufficiently small,*

$$\left\langle\left(L_+^{(m)} - \frac{b^2}{4}r^2\right)w, w\right\rangle \ge \delta_+\|w\|_{H^1}^2 - \frac{1}{\delta_+}\langle w, \phi_+\rangle^2,$$

for $\delta_+ > 0$ constant and any $w \in H^1_{(m)}$ vanishing at $r = (1-\eta)R_b$.

[11] The terms due to $R_b \neq \frac{2}{b}$ have no effect. Part of the error term $G_1(R)$ that appears in [18] has been moved to the left hand side of (A.107), so that the constant A_0 that appears in [18] can be ignored.

Lemma A.1 is analogous to [19, equation (212)], and is adapted from Lemma 2.15 by using a cutoff and the exponential decay of ϕ_+. Details can be found, [19, p660].

Step 6: Frechet Derivative on Fixed Domain

The aim is to prove that there exists,

$$\frac{\partial}{\partial_b} T_{b,b_0}\Big|_{b=b_0} \in H^1_{(m)}. \tag{A.108}$$

We will follow the argument of [18, p610], and revisit equation (A.107). In the limit $b \to b_0$ we have, with respect to L^2-norm,

$$\left(L_+^{(m)} - \frac{b_0^2}{4}r^2\right)\frac{T_\Delta}{b-b_0} = 0 - 3\left(\left(R^{(m)}\right)^2 - \left(P_{b_0}^{(m)}\right)^2\right)\frac{T_\Delta}{b-b_0} + \frac{\partial}{\partial_b}\mathcal{F}_{b,b_0}\Big|_{b=b_0}. \tag{A.109}$$

Note that by direct calculation,

$$\frac{\partial}{\partial_b}\mathcal{F}_{b,b_0}\Big|_{b=b_0} = \frac{2}{b_0}\left(1 - \frac{b_0^2}{4}r^2 - \frac{1}{2}\frac{\sqrt{1+b_0^2m^2}-1}{\sqrt{1+b_0^2m^2}}\right)P_{b_0}^{(m)},$$

and clearly exists. To show equation (A.108), we recall from Step 5 that, for $b_0 > 0$ sufficiently small, $L_+^{(m)} - \frac{b_0^2}{4}r^2$ is invertible over the subspace of $L^2_{(m)}$ functions that vanish at $r = (1-\eta)R_b$.

Step 7: Uniform Bound for $\partial_b T_{b,b_0}|_{b=b_0}$ (in C^2 with exponential weight)

Revisit equation (A.107), again in the limit $b \to b_0$ with respect to L^2 norm,

$$\left(L_+^{(m)} - \frac{b_0^2}{4}r^2 + 3\left(\left(R^{(m)}\right)^2 - \left(P_{b_0}^{(m)}\right)^2\right)\right)\frac{\partial}{\partial b}T_{b,b_0}\Big|_{b=b_0} = +\frac{\partial}{\partial_b}\mathcal{F}_{b,b_0}\Big|_{b=b_0}.$$

Similar to Step 4, we apply a maximum principle argument on the region $\frac{1}{\eta} < r \le (1-\eta)R_b$ to prove,

$$\| e^{(1-C\eta)R_b\Theta\left(\frac{r}{R_b}\right)}\frac{\partial}{\partial b}T_{b,b_0}\Big|_{b=b_0}\|_{C^2(r<(1-\eta)R_b)} \lesssim \frac{1}{b_0}.$$

The full argument is the same as [18, p610-611] with only minor adaptations.

Step 8: Uniform Bound for $\partial_b \widetilde{P}_b^{(m)}\Big|_{b=b_0}$ (in C^2 with exponential weight)

Let $\widetilde{P}_b^{(m)} = \phi_b P_b^{(m)}$ where ϕ_b are the smooth cutoff functions,

$$\phi_b(r) = \begin{cases} 1 \text{ for } r < (1-\eta)^2 R_b \\ 0 \text{ for } r > (1-\eta)R_b, \end{cases} \tag{A.110}$$

with the good behaviour, $\|\nabla\phi_b\|_{L^\infty} + \|\Delta\phi_b\|_{L^\infty} \to 0$, as $b \to 0$. Alternately,

$$\widetilde{P}_b^{(m)} = (\phi_b - \phi_{b_0})P_b^{(m)} + \phi_{b_0}\left(P_b^{(m)} - P_{b_0}^{(m)}\right) + \widetilde{P}_{b_0}^{(m)}. \tag{A.111}$$

The goal is to prove that,

$$\| e^{(1-C\eta)R_b\Theta\left(\frac{r}{R_b}\right)}\frac{\partial}{\partial_b}\widetilde{P}_b^{(m)}\|_{C^2(\mathbb{R}^2)} \longrightarrow 0 \quad \text{as} \quad b \to 0. \tag{A.112}$$

which is the second precursor to (2.22). Regarding the first right hand term of (A.111), we may re-express $P_b^{(m)}$ in terms of T_{b,b_0}. Then by Step 7 and the support of $\phi_b - \phi_{b_0}$, the contribution from that term is neglible. The remaining term, $\phi_{b_0}\left(P_b^{(m)} - P_{b_0}^{(m)}\right)$, is treated with calculations similar to those applied to T_Δ in Steps 5, 6 and 7. Details can be found, [18, p611-612].

Step 9: Supercritical Mass

The proof of (2.23) is due to [22, Lemma 1]. Here, we give a summary for the reader's convenience. To begin, note from equation (2.18) that $\widetilde{P}_b^{(m)}$ is formally a function of b^2. Then from Step 8 and the chain rule we conclude that, with an exponential weight, $\partial_{(b^2)}\widetilde{P}_b^{(m)}$ is bounded in C^2. From equation (2.18) it can be shown in the limit $b \to 0$ that,

$$L_+\frac{\partial}{\partial(b^2)}\widetilde{P}_b^{(m)} = \frac{r^2}{4}\widetilde{P}_b^{(m)}. \tag{A.113}$$

Consider then a product of (A.113) by $\Lambda R^{(m)}$,

$$\begin{aligned}\frac{1}{4}\int |x|^2|R^{(m)}|^2\,dx &= -\frac{1}{4}\left\langle r^2\,R^{(m)},\Lambda R^{(m)}\right\rangle\\ &= -\lim_{b\to 0}\left\langle L_+\partial_{(b^2)}P_b^{(m)},\Lambda R^{(m)}\right\rangle\\ &= -\lim_{b\to 0}\left\langle \partial_{(b^2)}P_b^{(m)},-2R^{(m)}\right\rangle \qquad = \lim_{b\to 0}\partial_{b^2}\|P_b^{(m)}\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

This concludes our summary of the proof of Proposition 2.1.

Proof of Proposition 2.3. Apply the point transformation, $e^{ib\frac{r^2}{4}}\zeta_b^{(m)} = e^{-im\theta}r^m Z(r)$. Then equation (2.52) reads,

$$\partial_r^2 Z + \frac{2m+1}{r}\partial_r Z - Z + \frac{b^2r^2}{4}Z = \widetilde{\Psi}_b,$$

where $r^m\widetilde{\Psi}_b = \Delta\phi_b P_b^{(m)} + \nabla\phi_b\cdot\nabla P_b^{(m)} + \left(\phi_b^3-\phi_b\right)P_b^{(m)}$. The arguments of [19, Appendix E] and [22, Appendix A], then prove a version of Proposition 2.3 for $e^{ib\frac{r^2}{4}}Z(r)$, as a radial function on $\mathbb{R}^{2m+2}$. By accounting for the equivalences of norms, this proves Proposition 2.3.

□

Appendix B. Details of Numerical Methods

Our numerical methods closely follow those detailed in [15], employing the Fortran 90/95 boundary value problem software described in [30]. We briefly review it here.

The software is designed to solve two point boundary value problems of the form

$$\frac{d}{dr}\mathbf{y} = \frac{1}{r}S\mathbf{y} + \mathbf{f}(r,\mathbf{y}), \tag{B.114}$$

by nonlinear collocation. Note that the algorithm handles r^{-1} singularities. All of our computations were performed on the domain $[0,50]$ with tolerance 10^{-10}.

Codes that can be used to reproduce the computations presented here are available at `http://www.math.toronto.edu/simpson/files/vortex_dist.tgz`.

B.1. **Point Transformations.** Unfortunately, the equation for the vortex state, (1.7), and the operators $\mathcal{L}_{1,\mathrm{rad}}^{(m)}$ and $\mathcal{L}_{2,\mathrm{rad}}^{(m)}$, include r^{-2} singularities. We address this with the point transformation

$$R^{(m)}(r) = r^m\tilde{R}^{(m)}(r). \tag{B.115}$$

Similarly, $U = e^{im\theta}r^m\tilde{U}^{(m)}(r)$ for any of the dependent variables. With this transformation, the vortex equation becomes,

$$(\tilde{R}^{(m)})'' + \frac{2m+1}{r}(\tilde{R}^{(m)})' - \tilde{R}^{(m)} + r^{2m}(\tilde{R}^{(m)})^3 = 0, \tag{B.116a}$$

$$(\tilde{R}^{(m)})'(0) = 0, \quad \lim_{r\to\infty}\tilde{R}^{(m)}(r) = 0, \tag{B.116b}$$

and the operators $\mathcal{L}_{1,\mathrm{rad}}^{(m)}$,$\mathcal{L}_{2,\mathrm{rad}}^{(m)}$ become,

$$\begin{aligned}\mathcal{L}_{1,\mathrm{rad}}^{(m)}U &= r^m\left\{-\tilde{U}'' - \frac{2m+1}{r}\tilde{U}' + 3r^{2m}\tilde{R}^{(m)}(m\tilde{R}^{(m)} + r(\tilde{R}^{(m)})')\tilde{U}\right\}\\ &= r^m\tilde{\mathcal{L}}_1\tilde{U}\end{aligned} \tag{B.117a}$$

$$\begin{aligned}\mathcal{L}_{2,\mathrm{rad}}^{(m)}Z &= r^m\left\{-\tilde{U}'' - \frac{2m+1}{r}\tilde{U}' + r^{2m}\tilde{R}^{(m)}(m\tilde{R}^{(m)} + r(\tilde{R}^{(m)})')\tilde{U}\right\}\\ &= r^m\tilde{\mathcal{L}}_2\tilde{Z}\end{aligned} \tag{B.117b}$$

The right hand sides of (3.90) conveniently become,

$$R^{(m)} = r^m \tilde{R}^{(m)} \tag{B.118}$$

$$\Lambda R^{(m)} = r^m \left\{(m+1)\tilde{R}^{(m)} + r(\tilde{R}^{(m)})'\right\} \tag{B.119}$$

$$\begin{aligned} \Lambda^2 R^{(m)} &= r^m \left\{(m+1)^2 \tilde{R}^{(m)} + (3+2m)r(\tilde{R}^{(m)})' + r^2(\tilde{R}^{(m)})''\right\} \\ &= r^m \left\{\left[(m+1)^2 + r^2\right] \tilde{R}^{(m)} + 2r(\tilde{R}^{(m)})' - r^{2(m+1)}(\tilde{R}^{(m)})^3\right\} \end{aligned} \tag{B.120}$$

B.2. **Artificial Boundary Conditions.** As the algorithm is designed to compute on finite intervals of $[a, b]$, we must compute on $[0, r_{\max}]$, where $r_{\max}$ is sufficiently large. This neccessitates the introduction of an artificial boundary condition on $\tilde{R}^{(m)}$, the vortex state, and U_j and Z_j solving the boundary value problems (3.90). The analogous question in the index function computations is verifying that there are no zeros beyond $r_{\max}$ which might have been missed.

To develop the artifiical boundary conditions, we examine the asymptotic behaviour of the solutions, using that potential terms are exponentially decaying. For the vortex state,

$$\tilde{R}^{(m)}(r) \propto r^{-m-\frac{1}{2}} e^{-r} \tag{B.121}$$

This gives us the boundary condition at $r_{\max}$

$$(\tilde{R}^{(m)})'(r_{\max}) + \left(1 + \frac{2m+1}{2r_{\max}}\right) \tilde{R}^{(m)}(r_{\max}) = 0 \tag{B.122}$$

which is accurate to $\mathrm{O}(r_{\max}^{-2})$.

By similar analysis the solutions to the linear boundary value problems, generically denoted by W, are

$$W(r) \propto r^{-2m} \tag{B.123}$$

as $r \to \infty$. Thus

$$\tilde{W}'(r_{\max}) + \frac{2m}{r_{\max}} \tilde{W}(r_{\max}) = 0 \tag{B.124}$$

This too is accurate to $\mathrm{O}(r_{\max}^{-2})$.

B.3. **Verification of Results.** With these approximations, we solve the following sets of equations, as single first order systems:

- The vortex $\tilde{R}^{(m)}$, and the index functions U and Z,
- The vortex $\tilde{R}^{(m)}$, the boundary value problem solutions U_1 and U_2, and the K_j inner products.
- The vortex $\tilde{R}^{(m)}$, the boundary value problem solutions Z_1 and Z_2, and the J_j inner products.

In computing the index functions, or alternatively the inner products, we are actually solving mixed initial value/boundary value problems.

We now present several *a postiori* checks on the accuracy of our results. All are based on checking that the behaviour of the solutions for large r is consistent with the anticipated asymptotic behavior.

B.3.1. *Verification of the Vortex States.* Two related ways of checking that we have adequately computed the vortex states are to examine its decay as r becomes large and to see that (B.121) becomes small as $r \to \infty$. For the vortices appearing in Figure 2, we plot these two metrics in Figures 4 and 5. With this artificial boundary condition, the exponential decay is well captured.

B.3.2. *Verification of the Index Count.* In counting the zeros of the index functions from Figure 3, there is the concern that there may be another root located beyond $r_{\max}$. To assess this, we note that the asympotically free behavior of $\tilde{U}$ and $\tilde{Z}$ is

$$\tilde{U}^{(m)} \sim C_0^{(m)} + C_1^{(m)} r^{-2m} \tag{B.125a}$$

$$\tilde{Z}^{(m)} \sim D_0^{(m)} + D_1^{(m)} r^{-2m} \tag{B.125b}$$

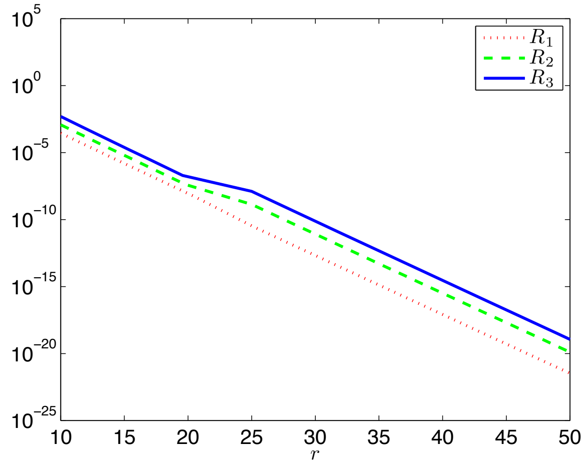


FIGURE 4. Behavior of the computed vortices as r becomes large. We recover the exponential decay.

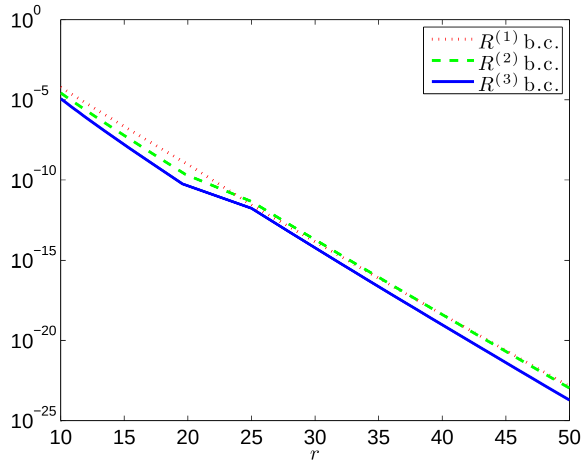


FIGURE 5. Asymptotically, the artificial boundary condition on the vortex is well satisfied.

We can estimate these constants by noting

$$\frac{\tilde{U}' r^{1+2m}}{-2m} \sim C_1^{(m)} \tag{B.126a}$$

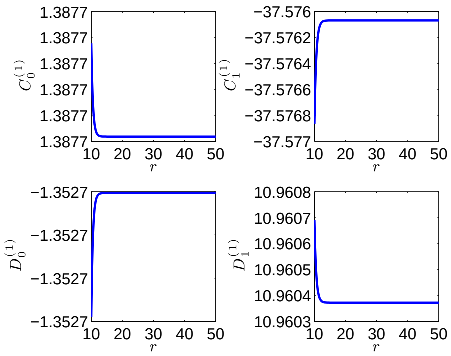
$$\tilde{U} + \frac{\tilde{U}' r}{2m} \sim C_0^{(m)} \tag{B.126b}$$

$$\frac{\tilde{Z}' r^{1+2m}}{-2m} \sim D_1^{(m)} \tag{B.126c}$$

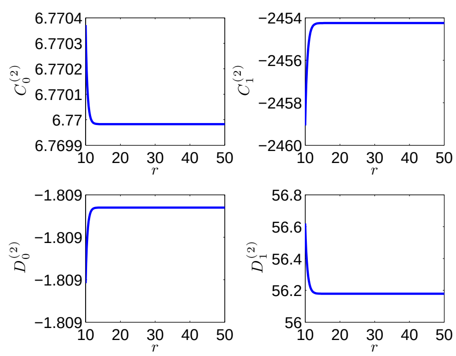
$$\tilde{Z} + \frac{\tilde{Z}' r}{2m} \sim D_0^{(m)} \tag{B.126d}$$

These constants are plotted in Figure 6. As they show, we have certainly computed into the "free" equation regime. More importantly, since $C_0^{(m)} > 0$ and $D_0^{(m)} < 0$ in all cases, we should not expect any additional zeros in the $U^{(m)}$ or $Z^{(m)}$ functions appearing in Figure 3.

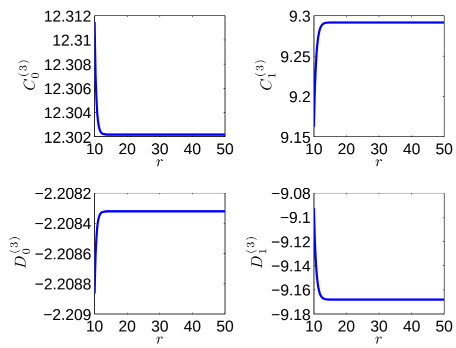
B.3.3. *Verification of the Inner Products.* For the inner product computations, we verify that in solving the boundary value problems, $U_l^{(m)}$, $Z_l^{(m)}$ adequately satisfy the artificial boundary conditions (B.124), and that



(a) $m = 1$



(b) $m = 2$



(c) $m = 3$

FIGURE 6. The asymptotical index function constants for winding numbers $m = 1, 2, 3$.

the K_l, J_l values are "constant". The check on the boundary conditions is given in Figures 7 and 8. As these figures show, (B.124) is well approximated.

In computing the inner products, we define

$$k_1^{(m)}(r) \equiv \int_0^r U_1^{(m)} R_1^{(m)} r dr. \tag{B.127}$$

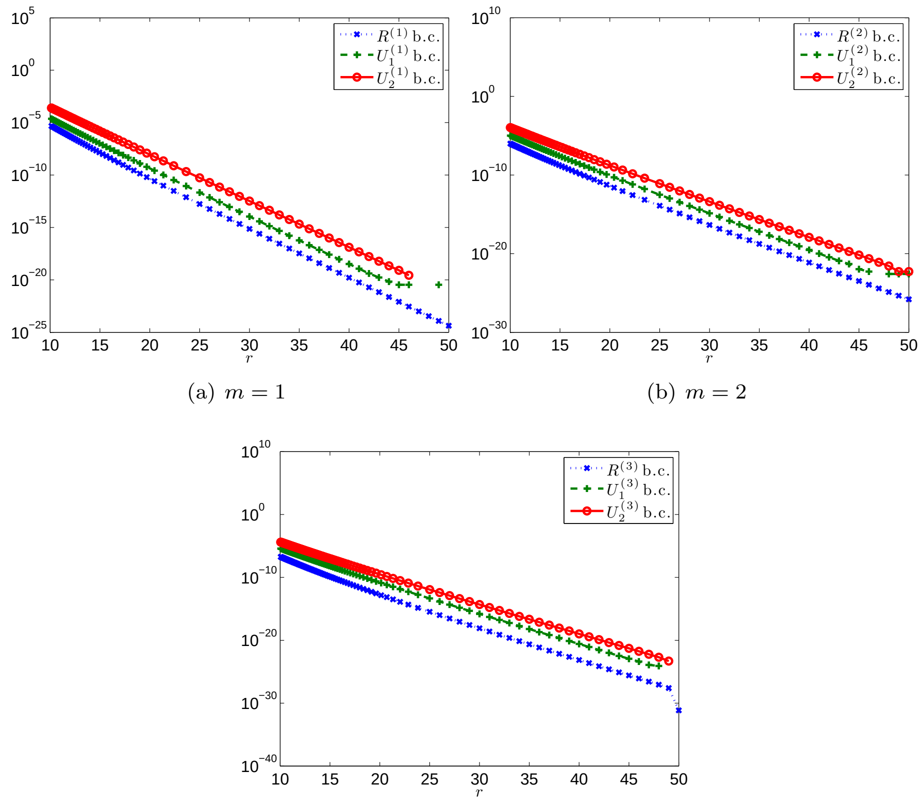
$k_2^{(m)}$, $k_3^{(m)}$, $j_1^{(m)}$, $j_2^{(m)}$, and $j_3^{(m)}$ are defined analogously. Clearly,

$$\lim_{r \to \infty} k_1^{(m)}(r) = K_1^{(m)} \tag{B.128}$$

and analogously for the other inner product values. We approximate,

$$K_1^{(m)} \approx k_1^{(m)}(r_{\max}), \tag{B.129}$$

for $r_{\max}$ sufficiently large that these converge to their limiting values. As Figures 9 and 10 show, this is indeed the case.



(a) $m = 1$

(b) $m = 2$

(c) $m = 3$

FIGURE 7. Check of the artificial boundary conditions on the $U_l^{(m)}$ functions.

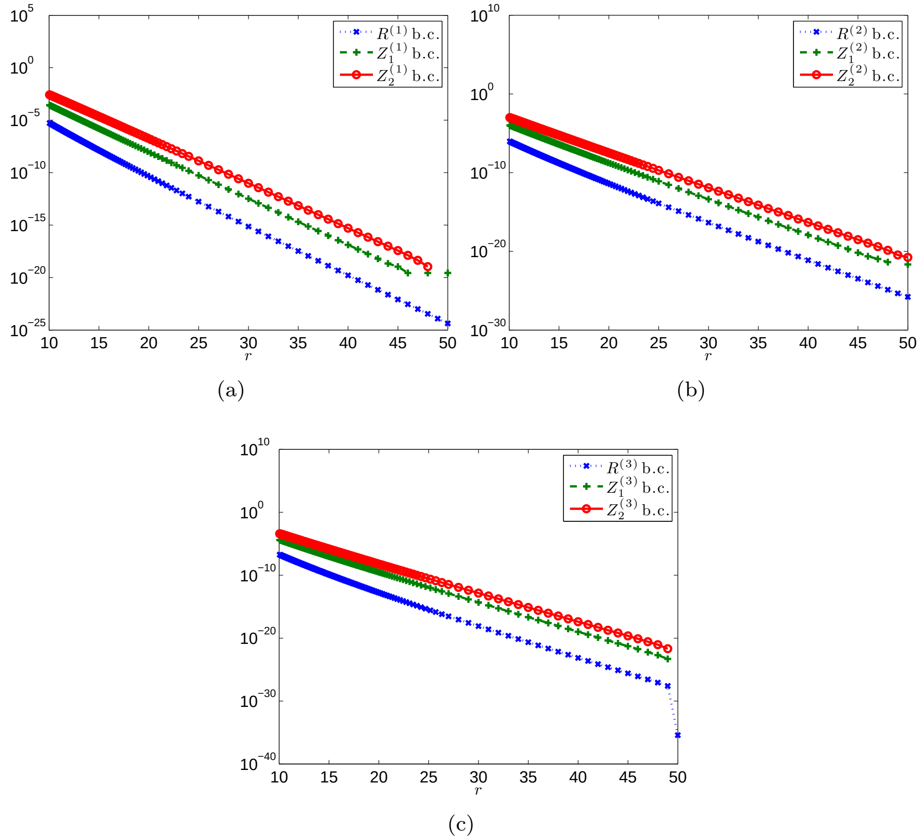
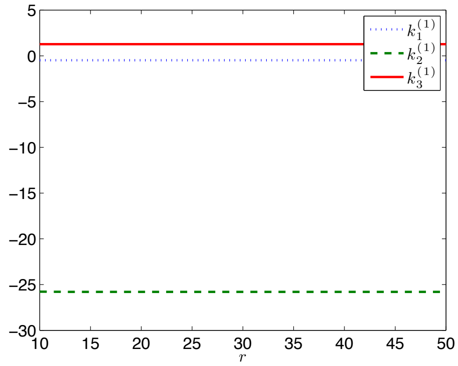
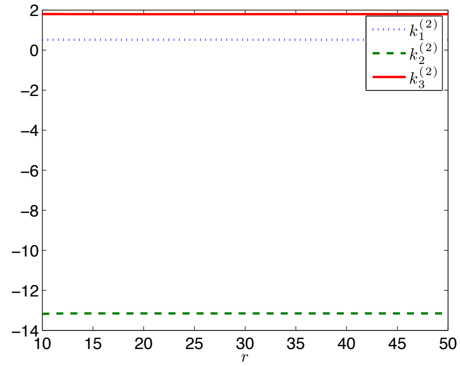


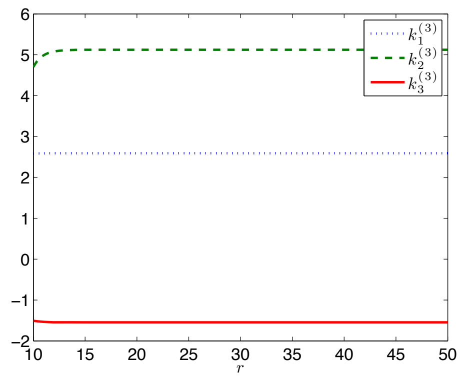
FIGURE 8. Check of the artificial boundary conditions on the $Z_l^{(m)}$ functions.



(a) $m = 1$

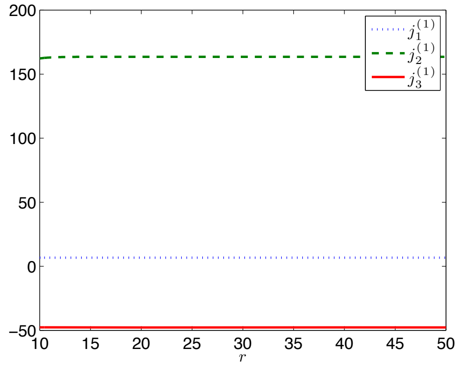


(b) $m = 2$

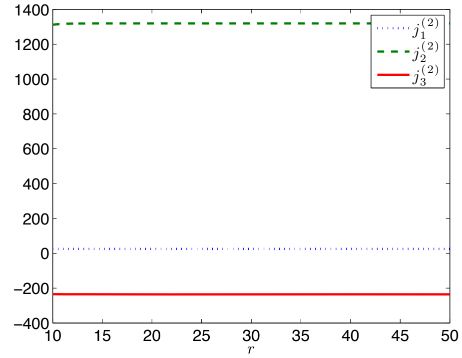


(c) $m = 3$

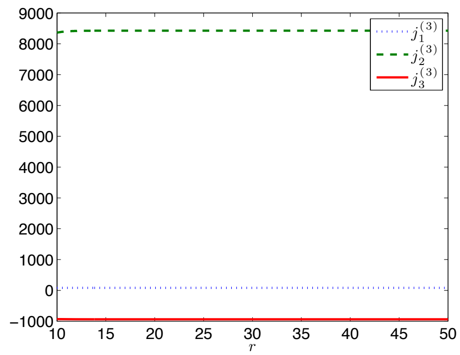
FIGURE 9. Convergence of the approximate inner product values to their limiting states.



(a) $m = 1$



(b) $m = 2$



(c) $m = 3$

FIGURE 10. Convergence of the approximate inner product values to their limiting states.

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